Math 2280 - Assignment 2

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Summer 2013

Section 1.5 - 1, 15, 21, 29, 38, 42

Section 1.6 - 1, 3, 13, 16, 22, 26, 31, 36, 56

Section 2.1 - 1, 8, 11, 16, 29

Section 2.2 - 1, 10, 21, 23, 24

Section 1.5 - Linear First-Order Equations

1.5.1 Find the solution to the initial value problem

$$y' + y = 2 \qquad y(0) = 0$$

1.5.15 Find the solution to the initial value problem

$$y' + 2xy = x, \qquad y(0) = -2.$$

1.5.21 Find the solution to the initial value problem

$$xy' = 3y + x^4 \cos x, \qquad y(2\pi) = 0.$$

1.5.29 Express the general solution of $dy/dx = 1 + 2xy$ in terms of the *error function*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

1.5.38 Consider the *cascade* of two tanks shown below with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volumes of brine in the two tanks. Each tank also initially contains 50 lbs of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank 1.
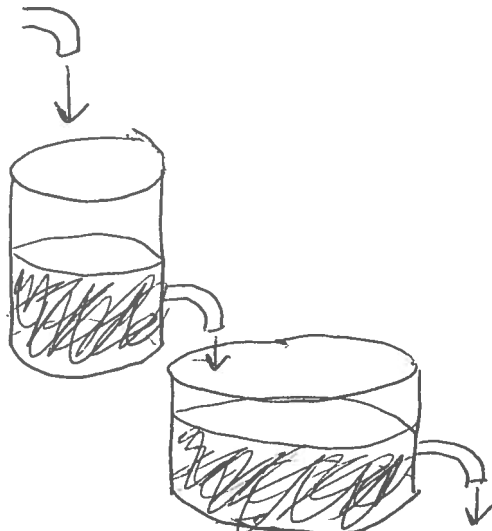
(a) Find the amount $x(t)$ of salt in tank 1 at time t.

(b) Suppose that $y(t)$ is the amount of salt in tank 2 at time t. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}.$$

and then solve for $y(t)$, using the function $x(t)$ found in part (a).

(c) Finally, find the maximum amount of salt ever in tank 2.

1.5.42 Suppose that a falling hailstone with density $\delta = 1$ starts from rest with negligible radius $r = 0$. Thereafter its radius is $r = kt$ (k is a constant) as it grows by accreation during its fall. Use Newton's second law - according to which the net force F acting on a possibly variable mass m equals the time rate of change dp/dt of its momentum $p = mv$ - to set up and solve the initial value problem

$$\frac{d}{dt}(mv) = mg, \qquad v(0) = 0,$$

where m is the variable mass of the hailstone, $v = dy/dt$ is its velocity, and the positive y-axis points downward. Then show that $dv/dt = g/4$. Thus the hailstone falls as though it were under *one-fourth* the influence of gravity.

Section 1.6 - Substitution Methods and Exact Equations

1.6.1 Find the general solution of the differential equation

$$(x + y)y' = x - y$$

1.6.3 Find the general solution of the differential equation

$$xy' = y + 2\sqrt{xy}$$

1.6.13 Find the general solution of the differential equation

$$xy' = y + \sqrt{x^2 + y^2}$$

Hint - You may find the following integral useful:

$$\int \ln \left(v + \sqrt{1 + v^2}\right) = \ln x + C.$$

1.6.16 Find the general solution of the differential equation

$$y' = \sqrt{x + y + 1}$$

1.6.22 Find the general solution of the differential equation

$$x^2 y' + 2xy = 5y^4$$

1.6.26 Find the general solution of the differential equation

$$3y^2y' + y^3 = e^{-x}$$

1.6.31 Verify that the differential equation

$$(2x + 3y)dx + (3x + 2y)dy = 0$$

is exact; then solve it.

1.6.36 Verify that the differential equation

$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$$

is exact; then solve it.

1.6.56 Suppose that $n \neq 0$ and $n \neq 1$. Show that the substitutuion $v = y^{1-n}$ transforms the Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

into the linear equation

$$\frac{dv}{dx} + (1-n)P(x)v(x) = (1-n)Q(x).$$

Section 2.1 - Population Models

2.1.1 Separate variables and use partial fractions to solve the initial value problem:

$$\frac{dx}{dt} = x - x^2 \qquad\qquad x(0) = 2.$$

2.1.8 Separate variables and use partial fractions to solve the initial value problem:

$$\frac{dx}{dt} = 7x(x - 13) \qquad\qquad x(0) = 17.$$

More space, if necessary, for problem 2.1.8.

2.1.11 Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$.

(a) Show that

$$P(t) = \left(\frac{1}{2}kt + \sqrt{P_0} \right)^2.$$

(b) If $P_0 = 100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

More space, if necessary, for problem 2.1.11.

2.1.16 Consider a rabbit population $P(t)$ satisfying the logistic equation $dP/dt = aP - bP^2$. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occuring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M?

More space, if necessary, for problem 2.1.16.

2.1.29 During the period from 1790 to 1930 the U.S. population $P(t)$ (t in years) grew from 3.9 million to 123.2 million. Throughout this period, $P(t)$ remained close to the solution of the initial value problem

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2, \qquad\qquad P(0) = 3.9.$$

(a) What 1930 population does this logistic equation predict?

(b) What limiting population does it predict?

(c) Has this logistic equation continued since 1930 to accurately model the U.S. population?

[This problem is based on the computation by Verhulst, who in 1845 used the 1790-1840 U.S. population data to predict accurately the U.S. population through the year 1930 (long after his own death, of course).]

More space, if necessary, for problem 2.1.29.

Section 2.2 - Equilibrium Solutions and Stability

2.2.1 - Find the critical points of the autonomous equation

$$\frac{dx}{dt} = x - 4.$$

Then analyze the sign of the equation to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation. Next, solve the differential equation explicitly for $x(t)$ in terms of t. Finally, use either the exact solution or a computer-generated slope field to sketch typical solution curves for the given differential equation, and verify visually the stability of each critical point.

More space, if necessary, for problem 2.2.1.

2.2.10 Find the critical points of the autonomous equation

$$\frac{dx}{dt} = 7x - x^2 - 10.$$

Then analyze the sign of the equation to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation. Next, solve the differential equation explicitly for $x(t)$ in terms of t. Finally, use either the exact solution or a computer-generated slope field to sketch typical solution curves for the given differential equation, and verify visually the stability of each critical point.

More space, if necessary, for problem 2.2.10.

2.2.21 Consider the differential equation $dx/dt = kx - x^3$.

(a) If $k \leq 0$, show that the only critical value $c = 0$ of x is stable.

(b) If $k > 0$, show that the critical point $c = 0$ is now unstable, but that the critical points $c = \pm\sqrt{k}$ are stable. Thus the qualitative nature of the solutions changes at $k = 0$ as the parameter k increases, and so $k = 0$ is a bifurcation point for the differential equation with parameter k.

The plot of all points of the form (k, c) where c is a critical point of the equation $x' = kx - x^3$ is the "pitchform diagram" show in figure 2.2.13 of the textbook.

More space, if necessary, for problem 2.2.21.

2.2.23 Suppose that the logistic equation $dx/dt = kx(M - x)$ models a population $x(t)$ of fish in a lake after t months during which no fishing occurs. Now suppose that, because of fishing, fish are removed from the lake at a rate of hx fish per month (with h a positive constant). Thus fish are "harvested" at a rate proportional to the existing fish population, rather than at the constant rate of Example 4 from the textbook.

(a) If $0 < h < kM$, show that the population is still logistic. What is the new limiting population?

(b) If $h \geq kM$, show that $x(t) \rightarrow 0$ as $t \rightarrow \infty$, so the lake is eventually fished out.

More space, if necessary, for problem 2.2.23.

2.2.24 Separate variables in the logistic harvesting equation

$$dx/dt = k(N - x)(x - H)$$

and then use partial fractions to derive the solution given in equation 15 of the textbook (also appearing in the lecture notes).

More space, if necessary, for problem 2.2.24.