Math 2280 - Assignment 1

Dylan Zwick

Summer 2013

Section 1.1 - 1, 12, 15, 20, 45

Section 1.2 - 1, 6, 11, 15, 27, 35, 43

Section 1.3 - 1, 6, 9, 11, 15, 21, 29

Section 1.4 - 1, 3, 17, 19, 31, 35, 53, 68

Section 1.1 - Differential Equations and Mathematical Models

1.1.1 Verify by substitution that the given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

$$y' = 3x^2; \qquad y = x^3 + 7$$

1.1.12 Verify by substitution that the given function is a solution of the given differential equation.

$$x^2y'' - xy' + 2y = 0; \qquad y_1 = x\cos(\ln x), \quad y_2 = x\sin(\ln x).$$

1.1.15 Substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation

$$y'' + y' - 2y = 0$$

1.1.20 First verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition.

$$y' = x - y; \qquad y(x) = Ce^{-x} + x - 1, \qquad y(0) = 10$$

1.1.45 Suppose a population P of rodents satisfies the differential equation $dP/dt = kP^2$. Initially, there are $P(0) = 2$ rodents, and their number is increasing at the rate of $dP/dt = 1$ rodent per month when there are $P = 10$ rodents. How long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?

More room for Problem 1.1.45, if you need it.

Section 1.2 - Integrals as General and Particular Solutions

1.2.1 Find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

$$\frac{dy}{dx} = 2x + 1; \qquad y(0) = 3.$$

1.2.6 Find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

$$\frac{dy}{dx} = x\sqrt{x^2 + 9} \qquad\qquad y(-4) = 0.$$

1.2.11 Find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

$$a(t) = 50,$$

$$v_0 = 10,$$

$$x_0 = 20.$$

1.2.15 Find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

$$a(t) = 4(t+3)^2,$$

$$v_0 = -1,$$

$$x_0 = 1.$$

1.2.27 A ball is thrown straight downward from the top of a tall building. The initial speed of the ball is $10m/s$. It strikes the ground with a speed of $60m/s$. How tall is the building?

1.2.35 A stone is dropped from rest at an initial height h above the surface of the earth. Show that the speed with which it strikes the ground is $v = \sqrt{2gh}$.

1.2.43 Arthur Clark's *The Wind from the Sun* (1963) describes Diana, a spacecraft propelled by the solar wind. Its aluminized sail provides it with a constant acceleration of $0.001g = 0.0098m/s^2$. Suppose this spacecraft starts from rest at time $t = 0$ and simultaneously fires a projectile (straight ahead in the same direction) that travels at one-tenth of the speed $c = 3 \times 10^8 m/s$ of light. How long will it take the spacecraft to catch up with the projectile, and how far will it have traveled by then?

13)

$y =$
axis

e of
alue

(14)

f the

(15)

origin

ider a
n Fig.

(16)

(17)

the left
hus the
y many

arantees
through
solution
nstance,
whole x-

1.3.1 and 1.3.6 See Below

In Problems 1 through 10, we have provided the slope field of the indicated differential equation, together with one or more solution curves. Sketch likely solution curves through the additional points marked in each slope field.

1. $\dfrac{dy}{dx} = -y - \sin x$

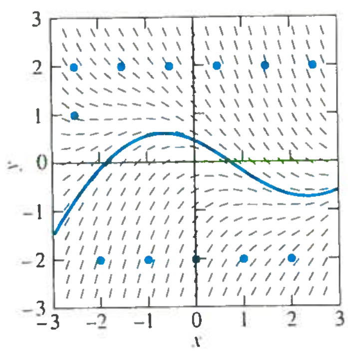

FIGURE 1.3.15.

2. $\dfrac{dy}{dx} = x + y$

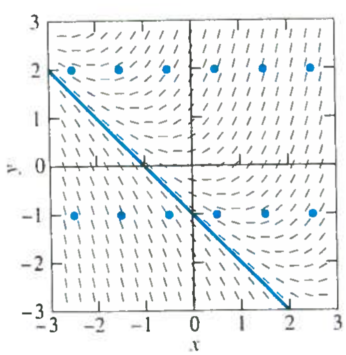

FIGURE 1.3.16.

3. $\dfrac{dy}{dx} = y - \sin x$

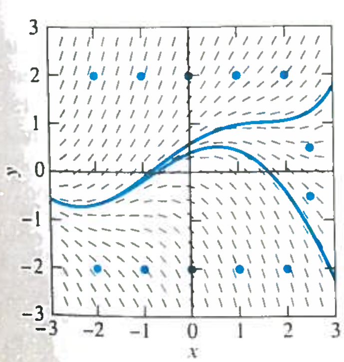

FIGURE 1.3.17.

4. $\dfrac{dy}{dx} = x - y$

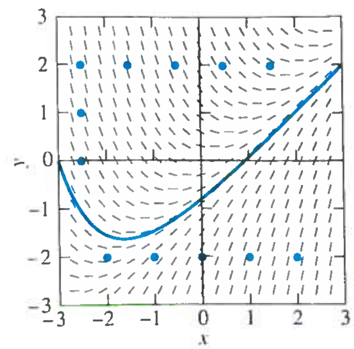

FIGURE 1.3.18.

5. $\dfrac{dy}{dx} = y - x + 1$

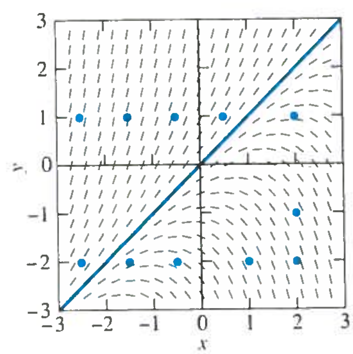

FIGURE 1.3.19.

6. $\dfrac{dy}{dx} = x - y + 1$

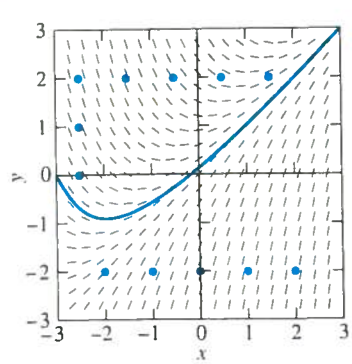

FIGURE 1.3.20.

1.3.9 See Below

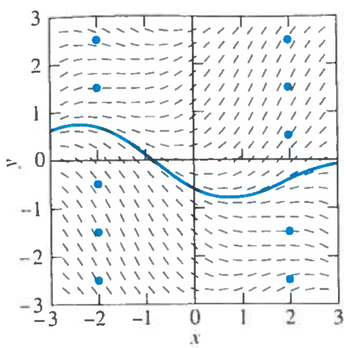

FIGURE 1.3.21.

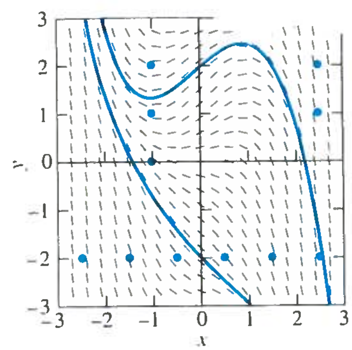

FIGURE 1.3.24.

8. $\dfrac{dy}{dx} = x^2 - y$

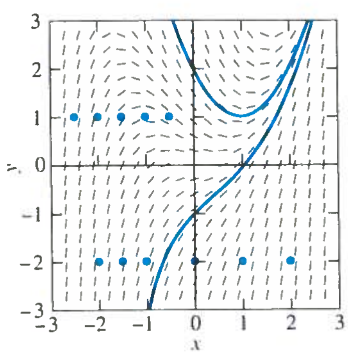

FIGURE 1.3.22.

9. $\dfrac{dy}{dx} = x^2 - y - 2$

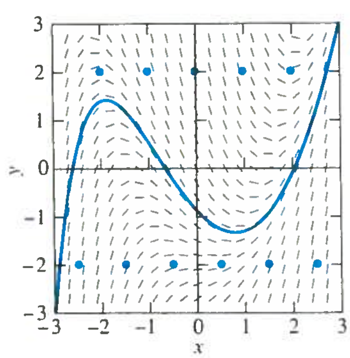

FIGURE 1.3.23.

In Problems 11 through 20, determine whether Theorem 1 does or does not guarantee existence of a solution of the given initial value problem. If existence is guaranteed, determine whether Theorem 1 does or does not guarantee uniqueness of that solution.

11. $\dfrac{dy}{dx} = 2x^2y^2; \quad y(1) = -1$

12. $\dfrac{dy}{dx} = x \ln y; \quad y(1) = 1$

13. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 1$

14. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 0$

15. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 2$

16. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 1$

17. $y\dfrac{dy}{dx} = x - 1; \quad y(0) = 1$

18. $y\dfrac{dy}{dx} = x - 1; \quad y(1) = 0$

19. $\dfrac{dy}{dx} = \ln(1 + y^2); \quad y(0) = 0$

20. $\dfrac{dy}{dx} = x^2 - y^2; \quad y(0) = 1$

In Problems 21 and 22, first use the method of Example 2 to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

21. $y' = x + y, \quad y(0) = 0; \quad y(-4) = ?$

22. $y' = y - x, \quad y(4) = 0; \quad y(-4) = ?$

1.3.11 Determine whether Theorem 1 does or does not guarantee existence of a solution of the given initial value problem. If existence is guaranteed, determine whether Theorem 1 does or does not guarantee uniqueness of that solution.

$$\frac{dy}{dx} = 2x^2y^2 \qquad y(1) = -1.$$

1.3.15 Determine whether Theorem 1 does or does not guarantee existence of a solution of the given initial value problem. If existence is guaranteed, determine whether Theorem 1 does or does not guarantee uniqueness of that solution.

$$\frac{dy}{dx} = \sqrt{x - y} \qquad y(2) = 2.$$

1.3.21 First use the method of Example 2 from the textbook to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

$$y' = x + y, \qquad y(0) = 0; \qquad y(-4) = ?$$

More room for Problem 1.3.21, if you need it.

1.3.29 Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & x \le c, \\ (x-c)^3 & x > c \end{cases}$$

satisfies the differential equation $y' = 3y^{\frac{2}{3}}$ for all x. Can you also use the "left half" of the cubic $y = (x-c)^3$ in piecing together a solution curve of the differential equation? Sketch a variety of such solution curves. Is there a point (a, b) of the xy-plane such that the initial value problem $y' = 3y^{\frac{2}{3}}, y(a) = b$ has either no solution or a unique solution that is defined for all x? Reconcile your answer with Theorem 1.

More room for Problem 1.3.29, if you need it.

Section 1.4 - Separable Equations and Applications

1.4.1 Find the general solution (implicit if necessary, explicit if convenient) to the differential equation

$$\frac{dy}{dx} + 2xy = 0$$

1.4.3 Find the general solution (implicit if necessary, explicit if convenient) to the differential equation

$$\frac{dy}{dx} = y \sin x$$

1.4.17 Find the general solution (implicit if necessary, explicit if convenient) to the differential equation

$$y' = 1 + x + y + xy.$$

Primes denote the derivatives with respect to x. (Suggestion: Factor the right-hand side.)

1.4.19 Find the explicit particular solution to the initial value problem

$$\frac{dy}{dx} = ye^x, \qquad y(0) = 2e.$$

1.4.31 Discuss the difference between the differential equations $(dy/dx)^2 = 4y$ and $dy/dx = 2\sqrt{y}$. Do they have the same solution curves? Why or why not? Determine the points (a, b) in the plane for which the initial value problem $y' = 2\sqrt{y}$, $y(a) = b$ has **(a)** no solution, **(b)** a unique solution, **(c)** infinitely many solutions.

More room for Problem 1.4.31, if you need it.

1.4.35 (Radiocarbon dating) Carbon extracted from an ancient skull contained only one-sixth as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

1.4.53 Thousands of years ago ancestors of the Native Americans crossed the Bering Strait from Asia and entered the western hemisphere. Since then, they have fanned out across North and South America. The single language that the original Native Americans spoke has since split into many Indian "language families." Assume that the number of these language families has been multiplied by 1.5 every 6000 years. There are now 150 Native American language families in the western hemisphere. About when did the ancestors of today's Native Americans arrive?

1.4.68 The figure below shows a bead sliding down a frictionless wire from point P to point Q. The *brachistochrone problem* asks what shape the wire should be in order to minimize the bead's time of descent from P to Q. In June of 1696, John Bernoulli proposed this problem as a public challenge, with a 6-month deadline (later extended to Easter 1697 at George Leibniz's request). Isaac Newton, then retired from academic life and serving as Warden of the Mint in London, received Bernoulli's challenge on January 29, 1697. The very next day he communicated his own solution - the curve of minimal descent time is an arc of an inverted cycloid - to the Royal Society of London. For a modern derivation of this result, suppose the bead starts from rest at the origin P and let $y = y(x)$ be the equation of the desired curve in a coordinate system with the y-axis pointing downward. Then a mechanical analogue of Snell's law in optics implies that

$$\frac{\sin\alpha}{v} = constant$$

where α denotes the angle of deflection (from the vertical) of the tangent line to the curve - so $\cot\alpha = y'(x)$ (why?) - and $v = \sqrt{2gy}$ is the bead's velocity when it has descended a distance y vertically (from $KE = \frac{1}{2}mv^2 = mgy = -PE$).

(a) First derive from $\sin\alpha/v = constant$ the differential equation

$$\frac{dy}{dx} = \sqrt{\frac{2a - y}{y}}$$

where a is an appropriate positive constant.

(b) Substitute $y = 2a \sin^2 t, dy = 4a \sin t \cos t dt$ in the above differential equation to derive the solution

$$x = a(2t - \sin 2t), \qquad y = a(1 - \cos 2t)$$

for which $t = y = 0$ when $x = 0$. Finally, the substitution of $\theta = 2a$ in the equations for x and y yields the standard parametric equations $x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$ of the cycloid that is generated by a point on the rim of a circular wheel of radius a as it rolls along the x-axis.