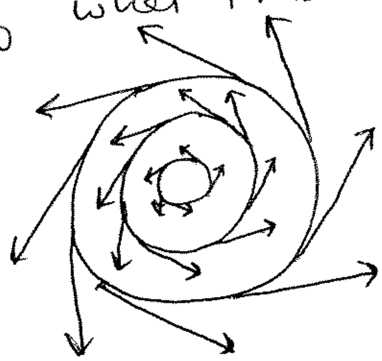
14.1 Vector Fields

So far, we're studied ① real-valued fns of one variable, ② vector-valued fns of one $\mathbb{R}$ variable, and ③ real-valued fns of several $\mathbb{R}$ variables. Now, it's time to look at vector-valued fns of several $\mathbb{R}$ variables. :)

vector field ⇒ a fn $\vec{F}$ that associates each vector input $\vec{p}$ (or a pt in n-space) with a vector output $\vec{F}$.

e.g. $\vec{F}(\vec{p}) = \vec{F}(x,y) = -y\hat{\imath} + x\hat{\jmath}$

We cannot really draw this, but we can draw a sample of what this may look like.



If $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ (i.e. the position vector), then

$\vec{F}(x,y)\cdot\vec{r} = (-y\hat{\imath}+x\hat{\jmath})\cdot(x\hat{\imath}+y\hat{\jmath}) = -xy+xy = 0$

$\Rightarrow \vec{r}$ is $\perp$ to $\vec{F} \Leftrightarrow \vec{F}$ vectors are $\perp$ to radius vectors or tangent to circles centered at origin (as drawn).

And $|\vec{F}| = \sqrt{y^2+x^2} = |\vec{r}|$, i.e. the magnitude of output vectors = radius of circles.

14.1 (continued)

Inverse Square law of Gravitational Attraction ⇒

(by Newton)

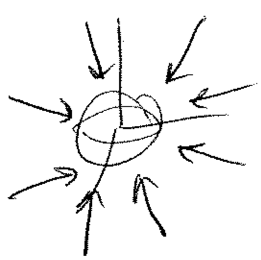
magnitude of force of attraction between objects of mass M + m is given by $\frac{GMm}{d^2}$, where d = distance between objects + G is universal constant.

i.e. $|\vec{F}| = \frac{GMm}{|\vec{r}|^2}$ $\quad$ $\vec{F}$ is "force field" (a vector field that represents a force) + $\vec{r}$ = radius vector from origin to pt.

Ex If we find gravitational force field $\vec{F}(x,y,z)$ exerted on mass in space (m) from earth (mass M), we can use above law.

$|\vec{F}| = \frac{GMm}{|\vec{r}|^2}$ but $\vec{F} = ?$ we know gravity pushes objects toward earth ⇒ the direction of $\vec{F}$ is opposite $\vec{r}$ ⇒ dir of $\vec{F} = \frac{-\vec{r}}{|\vec{r}|}$ (unit vector)

$$\Rightarrow \vec{F} = \frac{GMm}{|\vec{r}|^2}\left(\frac{-\vec{r}}{|\vec{r}|}\right) = \frac{-GMm}{|\vec{r}|^3}\vec{r}$$



14.1 (continued)

Conservative vector field $\Rightarrow$ $\vec{F}(x,y)$ is conservative if it is the gradient of a scalar field f. And f is then called its potential function.

(fields that obey the inverse square law are conservative!)

Defn

Let $\vec{F}(x,y,z) = M(x,y,z)\,\hat{\imath} + N(x,y,z)\,\hat{\jmath} + P(x,y,z)\,\hat{k}$ be a vector field $\ni$ $\frac{\partial M}{\partial x}, \frac{\partial N}{\partial y}, \frac{\partial P}{\partial z}$ exist. Then,

(scalar) $\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$ (divergence of $\vec{F}$)

(vector) $\operatorname{curl} \vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{\imath} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$

Notation If we let $\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$, then

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (M\hat{\imath} + N\hat{\jmath} + P\hat{k})$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \operatorname{div} \vec{F}$$

& $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \operatorname{curl} \vec{F}$

$$\boxed{\begin{aligned} \nabla \cdot \vec{F} &= \operatorname{div} \vec{F} \\ \nabla \times \vec{F} &= \operatorname{curl} \vec{F} \end{aligned}}$$

14.1 (continued)

Ex1 Find div $\vec{F}$ + curl $\vec{F}$.

(a) $\vec{F}(x,y,z) = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

(b) $F(x,y,z) = \cos x\,\hat{i} + \sin y\,\hat{j} + 3\hat{k}$

14.1 (continued)

Ex 1 (cont)

(c) $\vec{F}(x,y,z) = (y+2z)\hat{\imath} + (-x+3z)\hat{\jmath} + (x+y+z)\hat{k}$

Ex 2 Show that $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$. (Assume all partial derivatives exist and are continuous.)

14.1 (continued)

Ex 3 Show $\text{curl}(\text{grad} f) = \vec{0}$. (Again, assume all partials exist + are continuous.)

If $\vec{F}$ represents a velocity field for a fluid, the $\text{div}\,\vec{F}$ at a pt P measures tendency to diverge away from P (or toward) ($\text{div}\,\vec{F} > 0 \Rightarrow$ away, $\text{div}\,\vec{F} < 0 \Rightarrow$ toward).

And $\text{curl}\,\vec{F}$ picks out direction of axis about which fluid rotates most rapidly. Magnitude of $\text{curl}\,\vec{F}$ gives its speed (of rotation).

14.2 Line Integrals

Remember arc length from chp 6 (pg 295).

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ where $x = x(t)$ & $y = y(t)$ represents a plane curve (in 2d).

Now, we consider a 3d curve that we get by tracing $x(t)$ & $y(t)$ on a 3d surface $f(x,y)$.

line integral $\Rightarrow \int_C f(x,y)\, ds$ where C is the 2d curve constraint in the xy-plane & ds is the arc length "little bit." (We can also call this a "curve integral.")

C is given by $x = x(t)$ & $y = y(t)$, $t \in [a,b]$ where x' & y' exist & are continuous & $x^2 + y^2 \neq 0$.

C is positively oriented if its direction corresponds to t going from a to b.

$$\Rightarrow \boxed{\int_C f(x,y)\, ds = \int_a^b f(x,y) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

and

$$\boxed{\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt}$$

14.2 (continued)

Ex 1 Evaluate $\int_C xy^{2/5}\, ds$ where C given by

$x = \frac{1}{2}t \quad y = t^{5/2} \quad t \in [0,1]$.

Ex 2 $\int_C (x^2+y^2+z^2)\, ds \qquad C: x = 4\cos t \quad y = 4\sin t \quad z = 3t$

$0 \le t \le 2\pi$

Ex 3 $\int_C y^3\,dx + x^3\,dy \qquad C: x=2t,\ y=t^2-3,\ -2\le t\le 1.$

If $x=2t$, then $dx = 2\,dt$. Similarly, $dy = 2t\,dt$

$$\Rightarrow \int_C y^3\,dx + x^3\,dy = \int_{-2}^{1} (t^2-3)^3(2\,dt) + (2t)^3(2t\,dt)$$

$$= \int_{-2}^{1} \left[(t^6-9t^4+27t^2-27)(2) + 8t^3(2t)\right] dt$$

$$= \int_{-2}^{1} 2t^6-18t^4+54t^2-54+16t^4\ dt$$

Ex 4 $\int_C y\,dx + x\,dy \qquad C: y=x^2,\ 0\le x\le 1$

14.2 (continued)

Ex 5 $\int_C (x+y+z)\,dx + (2x-3y)\,dy + (3x-y+z)\,dz$

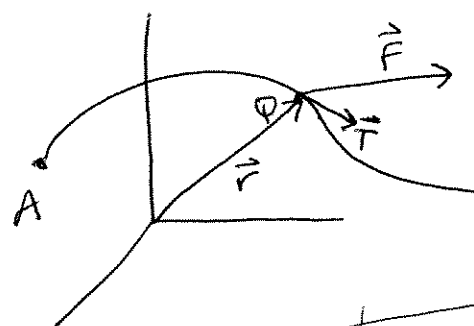
C is line segment path from $(0,0,0)$ to $(0,2,0)$ to $(3,2,0)$ to $(3,2,-1)$.

$\boxed{\text{Work}}$ $\vec{F}(x,y,z) = M(x,y,z)\hat{\imath} + N(x,y,z)\hat{\jmath} + P(x,y,z)\hat{k}$

is force at pt (x,y,z) ($M, N + P$ are continuous)

If we want to find work done by $\vec{F}$ in moving a particle along smooth, oriented curve C, then we can add up all the "little bits" of work done at each pt along the way. (Positive work means its done in direction of curve C.)

$\vec{T}$ = unit tangent vector $= \frac{d\vec{r}}{ds}$



$\vec{F} \cdot \vec{T}$ = tangential component of $\vec{F}$ (at Q)

work $\Rightarrow$
$$\boxed{\begin{aligned} W &= \int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt = \int_C \vec{F} \cdot d\vec{r} \\ &= \int_C M\,dx + N\,dy + P\,dz \end{aligned}}$$

14.3 Independence of Path

Fundamental Thm for line Integrals

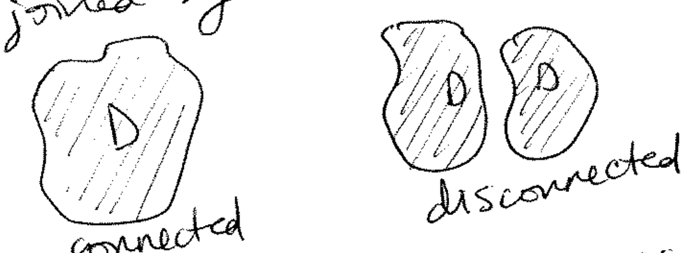
Let C be piecewise smooth curve given by $\vec{r}=\vec{r}(t)$ $t\in[a,b]$ which begins at $\vec{a}=\vec{r}(a)$ + ends at $\vec{b}=\vec{r}(b)$. If f is continuously differentiable on an open set containing C, then

$$\int_C \nabla f(\vec{r})\cdot d\vec{r} = f(\vec{b})-f(\vec{a})$$

Analog to 2nd Fundamental Thm of Calculus

$$\int_a^b \frac{df}{dx}\,dx = f(b)-f(a).$$

<u>connected set</u> $\Rightarrow$ a set D is connected if any 2 pts in D can be joined by a piecewise smooth curve lying entirely in D.



<u>independent of path</u> $\Rightarrow$ $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ is independent of path in D if for any 2 pts A + B in D, the line integral has the same value $\forall$ path C in D positively oriented from A to B.

14.3 (continued)

Independence of Path Thm

Let $\vec{F}(\vec{r})$ be continuous on an open connected set D. Then the line integral $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ is independent of path iff $\vec{F}(\vec{r}) = \nabla f(\vec{r})$ for some scalar functn f, i.e. $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ is independent of path iff $\vec{F}$ is a conservative vector field on D.

$\vec{F}(\vec{r}) = \nabla f(\vec{r})$

$\Rightarrow$ $\boxed{\int_C \vec{F}(\vec{r})\cdot d\vec{r} = 0}$ if C is a closed curve in D.

since $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ is independent of path

$\Rightarrow \vec{F}(\vec{r}) = \nabla f(\vec{r})$ + we know $\int_C \nabla f(\vec{r})\cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$

$= f(\vec{a}) - f(\vec{a}) = 0$

(if curve is closed, then $a = b$).

Three equivalent statements: (i.e. $\vec{F}$ is conservative)

① $\vec{F} = \nabla f$ for some fn f

② $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ is independent of path

③ $\int_C \vec{F}(\vec{r})\cdot d\vec{r} = 0$ $\forall$ closed path

(☆ Work done by a conservative force field around a closed path is zero.)

14.3 (continued)

Thm
Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ where $M, N + P$ are continuous and their 1st order partial derivatives are continuous on an open connected set D, which is also simply connected (i.e. no "holes" or "tunnels"). Then $\vec{F}$ is conservative iff $\text{curl}\,\vec{F} = \vec{0}$

i.e. $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, $\dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x}$ + $\dfrac{\partial N}{\partial z} = \dfrac{\partial P}{\partial y}$.

In 2d case, $\vec{F}$ is conservative iff $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Ex 1 Is $\vec{F} = (12x^2 + 3y^2 + 5y)\hat{i} + (6xy - 3y^2 + 5x)\hat{j}$ conservative?

14.3 (continued)

Ex 2 Using $\vec{F}$ from Ex 1, find $f \ni \vec{F} = \nabla f$.

$$\vec{F} = (12x^2 + 3y^2 + 5y)\hat{\imath} + (6xy - 3y^2 + 5x)\hat{\jmath}$$

$$\vec{F} = \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = M\hat{\imath} + N\hat{\jmath}$$

$\Rightarrow \dfrac{\partial f}{\partial x} = 12x^2 + 3y^2 + 5y$ We can antidifferentiate both sides wrt x

$\Rightarrow f = 4x^3 + 3y^2x + 5yx + C(y)$ where $C(y)$ is some fn of y.

and $\dfrac{\partial f}{\partial y} = 6xy - 3y^2 + 5x$. So if we differentiate our new found f wrt y, we get

$$\frac{\partial f}{\partial y} = 6yx + 5x + C'(y)$$

+ that must be $= 6xy - 3y^2 + 5x = N(x,y)$

$\Rightarrow 6xy + 5x + C'(y) = 6xy + 5x - 3y^2$

$\Leftrightarrow C'(y) = -3y^2 \Rightarrow C(y) = -y^3 + C_2$ where $C_2 \in \mathbb{R}$

$$\Rightarrow f(x,y) = 4x^3 + 3y^2x + 5yx + -y^3 + C$$

Ex 3 Using same $\vec{F} = (12x^2+3y^2+5y)\hat{\imath} + (6xy-3y^2+5x)\hat{\jmath}$,
calculate $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ where C is any path
from $(0,0)$ to $(2,1)$.

$$\int_C \vec{F}(\vec{r})\cdot d\vec{r} = \int_C \langle 12x^2+3y^2+5y,\ 6xy-3y^2+5x\rangle \cdot \langle dx, dy\rangle$$

$$= \int_C (12x^2+3y^2+5y)\,dx + (6xy-3y^2+5x)\,dy$$

14.3 (continued)

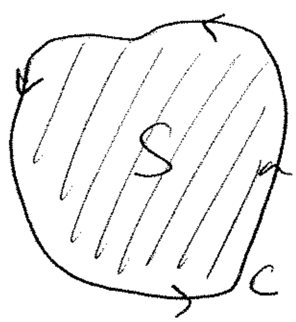
Ex 4 Show that the line integral

$$\int_{(0,1,0)}^{(1,1,1)} (yz+1)\,dx + (xz+1)\,dy + (xy+1)\,dz$$ is

independent of path + then evaluate the integral.

14.4 Green's Theorem in the Plane

$\oint_C M\,dx + N\,dy$ means the line integral of $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ around $C \ni C$ is a simple closed curve that forms the bndry of a planar region S (in xy-plane). And C is oriented so S is always to left.



Green's Thm

Let C be a piecewise smooth, simple closed curve that forms a bndry of a region S in the xy-plane. If $M+N$ are continuous w/ continuous partial derivatives on S + bndy C, then

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\,dx + N\,dy$$

★ If $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$, then $\oint_C M\,dx + N\,dy = 0 \Rightarrow \vec{F}$ is conservative.

Green's Thm is useful because it provides another way to do a line integral (for cases where the line integral is too ugly).

14.4 (continued)

Ex 1 Use Green's Thm to evaluate

$\oint_C \sqrt{y}\,dx + \sqrt{x}\,dy$ where C is the closed curve

formed by $y=0$, $x=2$ + $y=\frac{x^2}{2}$.

14.4 (continued)

If $M = \frac{-y}{2}$ & $N = \frac{x}{2}$, then

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_S \left(\frac{1}{2} - \frac{-1}{2}\right) dA = \iint_S dA$$

$$\Rightarrow \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \text{Area of region enclosed in } S. \; A(S)$$

We can apply Green's Thm,

$$A(S) = \oint_C M\,dx + N\,dy = \oint_C \frac{-y}{2}\,dx + \frac{x}{2}\,dy$$

$$\Leftrightarrow \boxed{A(S) = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

where C is bndry of S.

Ex 2 Use above result to derive area of circle with radius $= r$, i.e. $x^2 + y^2 = r^2$.

14.4 (continued)

Ex 3 Use $A(S) = \frac{1}{2}\oint_C x\,dy - y\,dx$ to find the area of S bndd by $y = \frac{1}{2}x^3$ & $y = x^2$.

14.4 (Continued)

Ex 4 (#19) Let $\vec{F} = \frac{y}{x^2+y^2}\hat{\imath} - \frac{x}{x^2+y^2}\hat{\jmath} = M\hat{\imath} + N\hat{\jmath}$

(a) Show that $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$.

(b) Calculate $\oint_C M\,dx + N\,dy$ where C is unit circle.

$x = \cos\theta \qquad dx = -\sin\theta\, d\theta$

$y = \sin\theta \qquad dy = \cos\theta\, d\theta$

(c) Why doesn't the result in (b) contradict Green's Thm?

14.5 Surface Integrals

If we have a fn $g(x,y,z)$ defined on some surface $f(x,y)$ + we want to add up lots of "little bits" of g on the surface, then we get a surface integral.

Thm

Let G be a surface given by $z=f(x,y)$, where (x,y) is in R (a closed + bndd set in xy-plane). If f has continuous 1st order partial derivatives + $g(x,y,z)=g(x,y,f(x,y))$ is continuous on R, then

$$\iint_G g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\sqrt{f_x^2+f_y^2+1}\,dy\,dx$$

Ex 1 Evaluate $\iint_G g(x,y,z)\,dS$ given $g(x,y,z)=x$ + G is on $x+y+2z=4$ $\ni$ $x\in[0,1]$ + $y\in[0,1]$.

14.5 (continued)

Ex 2 Evaluate $\iint_G 2y^2 + z \, dS$ given G is surface $z = x^2 - y^2$, $0 \le x^2 + y^2 \le 1$.

14.5 (continued)

Ex 3 Evaluate $\iint_G g(x,y,z)\,dS$ if $g(x,y,z) = z$ & G is the tetrahedron bndd by the coordinate planes and the plane $4x+8y+2z=16$.

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14.4 Vector Forms of Green's Theorem

(1)

C = a smooth, simple closed curve in xy-plane

S = region inside C

$\vec{T}$ = unit tangent vector to C

$\vec{n}$ = unit normal vector to C

If $x = x(s)$ + $y = y(s)$ (i.e. x + y are given parametrically), then

$$\vec{T} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}$$

and

$$\vec{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$$

$> \vec{T} \perp \vec{n}$

If $\vec{F}(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$, then

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C (M\hat{i} + N\hat{j})\cdot\left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right) ds$$

$$= \oint_C \left(M\frac{dy}{ds} + -N\frac{dx}{ds}\right) ds$$

$$= \oint_C M\,dy - N\,dx = \oint_C -N\,dx + M\,dy$$

and by Green's Theorem, this implies

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_S \frac{\partial(M)}{\partial x} - \frac{\partial(-N)}{\partial y}\,dA$$

$$= \iint_S \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dA$$

But $\nabla\cdot\vec{F} = \langle \partial/\partial x, \partial/\partial y\rangle \cdot \langle M, N\rangle = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$

$$\Rightarrow \boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_S \nabla\cdot\vec{F}\,dA}$$

This represents flux of vector field $\vec{F}$ across boundary C (in outward direction)

14.4 (continued)

(The flux basically measures the amount of fluid leaving S due to the vector field $\vec{F}$.)

(2) Now, assume we can write our 2d vector field as a 3d vector field, w/ the z-component $=0$. That is, we have the same S + C (2d) embedded in a 3d coordinate system.

So, $\vec{F}(x,y) = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath} + 0\hat{k}$

and $\vec{T} = \frac{dx}{ds}\hat{\imath} + \frac{dy}{ds}\hat{\jmath} + 0\hat{k}$

$\Rightarrow \vec{F}\cdot\vec{T} = \langle M, N, 0\rangle \cdot \langle \frac{dx}{ds}, \frac{dy}{ds}, 0\rangle = M\frac{dx}{ds} + N\frac{dy}{ds}$

$\Rightarrow \oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy$

Using Green's Theorem, we get

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$$

Notice that $\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & 0 \end{vmatrix}$

$= \hat{\imath}\left(-\frac{\partial N}{\partial z}\right) - \hat{\jmath}\left(-\frac{\partial M}{\partial z}\right) + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$= 0\hat{\imath} + 0\hat{\jmath} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$

since $M = M(x,y)$ + $N = N(x,y) \Rightarrow \frac{\partial M}{\partial z} = \frac{\partial N}{\partial z} = 0$

$\Rightarrow \operatorname{curl}\vec{F}\cdot\hat{k} = \langle 0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\rangle \cdot \langle 0, 0, 1\rangle$

$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

14.4 (continued)

$$\Longleftrightarrow \quad \oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA$$

$$\Longleftrightarrow \quad \boxed{\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S \operatorname{curl}\vec{F}\cdot\hat{k}\,dA}$$

Altogether, we now have 3 forms of Green's Thm

$$① \quad \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\,dx + N\,dy$$

$$② \quad \iint_S (\nabla\cdot\vec{F})\,dA = \oint_C \vec{F}\cdot\vec{n}\,ds$$

and

$$③ \quad \iint_S (\operatorname{curl}\vec{F})\cdot\hat{k}\,dA = \oint_C \vec{F}\cdot\vec{T}\,ds$$

14.6/14.7 Gauss + Stokes Theorems

Notation: ∂S refers to the boundary of S + S is a closed + bounded region (in either 2d or 3d space)

(We used to call C the boundary curve of S in 2d.)

We know from Green's Thms that

$$\oint_{\partial S} \vec{F}\cdot\vec{n}\, ds = \iint_S \operatorname{div}\vec{F}\, dA \quad \text{for } \vec{F} \text{ in 2d.}$$

Let's extend this to 3d. ∂S is ~~boundary~~ boundary of a closed, bounded solid S in 3d. (∂S is surface.)

Gauss' Thm

Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ be a vector field $\ni$ M, N + P have continuous 1^{st} order partial derivatives on a solid S w/ boundary ∂S. If $\vec{n}$ denotes the outer unit normal vector to ∂S, then

$$\iint_{\partial S} \vec{F}\cdot\vec{n}\, dS = \iiint_S \operatorname{div}\vec{F}\, dV$$

(2) generalize

We can write $\vec{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ where α, β + γ are direction angles for $\vec{n}$.

$$\Rightarrow \vec{F}\cdot\vec{n} = \langle M, N, P\rangle \cdot \langle \cos\alpha, \cos\beta, \cos\gamma\rangle$$
$$= M\cos\alpha + N\cos\beta + P\cos\gamma$$

14.6/14.7 (continued)

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S (M\cos\alpha + N\cos\beta + P\cos\gamma)\, dS$$

$$\Rightarrow \iint_S (M\cos\alpha + N\cos\beta + P\cos\gamma)\, dS = \iiint_S \text{div}\,\vec{F}\, dV$$

$$\Leftrightarrow \boxed{\iint_S (M\cos\alpha + N\cos\beta + P\cos\gamma)\, dS = \iiint_S \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dV}$$

Alternate Form of Gauss' Thm

Now, let's generalize form ③ of Green's Theorem!

Stoke's Theorem

Let S be a two-sided surface w/ a continuously varying unit normal $\vec{n}$, ∂S is the piecewise smooth, simple closed curve that forms the boundary of S (oriented positively). Suppose $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is a vector field w/ $M, N, + P$ having continuous 1st order partial derivatives on S + its boundary ∂S. If $\vec{T}$ denotes the unit tangent vector to ∂S, then

$$\oint_{\partial S} \vec{F}\cdot\vec{T}\, ds = \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\, dS$$