Math2210 Midterm 3 Summer, 2012 Dylan Zwick

Name Key Date 7-26-2012

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. Evaluate the integrals.

(a) (10 points) $\int_1^2 \int_0^3 xy + y^2 \, dy\, dx$.

$$\int_1^2 \frac{xy^2}{2} + \frac{y^3}{3} \Big|_0^3 dx = \int_1^2 \left(\frac{9}{2}x + 9\right) dx$$

$$= \frac{9}{4}x^2 + 9x \Big|_1^2 = (9+18) - \left(\frac{9}{4}+9\right) = 18 - \frac{9}{4}$$

$$= \frac{72}{4} - \frac{9}{4} = \frac{63}{4}$$

Answer 1(a): $\boxed{63/4}$

(b) (15 points) $\int_0^{\pi/2} \int_{\sin 2z}^{0} \int_0^{2yz} \sin\left(\frac{x}{y}\right) dx\, dy\, dz$.

$$= \int_0^{\pi/2} \int_{\sin(2z)}^{0} -y\cos\left(\frac{x}{y}\right) \Big|_0^{2yz} dy\, dz$$

$$= \int_0^{\pi/2} \int_{\sin(2z)}^{0} \left(-y\cos(2z) + y\right) dy\, dz$$

$$= \int_0^{\pi/2} \frac{-y^2\cos(2z)}{2} + \frac{y^2}{2} \Big|_{\sin(2z)}^{0} dz = \int_0^{\pi/2} \frac{\sin^2(2z)(-1+\cos(2z))}{2} dz$$

$$= -\int_0^{\pi/2} \frac{\sin^2(2z)}{2} dz + \int_0^{\pi/2} \frac{\cos(2z)\sin^2(2z)}{2} dz \leftarrow \begin{array}{l} u = \sin(2z) \\ du = 2\cos(2z)\,dz \end{array}$$

$$= \frac{1}{4}\int_0^0 u^2\, du = 0$$

$$= -\frac{1}{4}\int_0^{\pi/2} (1 - \cos(4z))\, dz$$

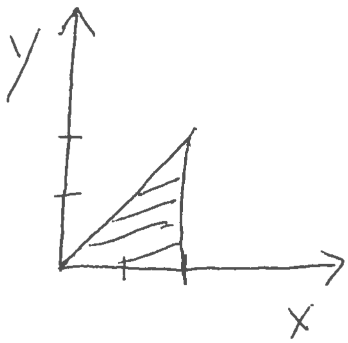
$$= -\frac{1}{4}\left(z - \frac{\sin(4z)}{4}\right)\Big|_0^{\pi/2}$$

Answer 1(b): $\boxed{-\frac{\pi}{8}}$

$$= -\frac{\pi}{8}$$

(Note: This is #1 continued!)

(c) (20 points) $\int_0^2 \int_y^2 e^{-x^2}\,dx\,dy$



Switching ~~limits~~ order of integration:

$$\int_0^2 \int_0^x e^{-x^2}\,dy\,dx = \int_0^2 x e^{-x^2}\,dx$$

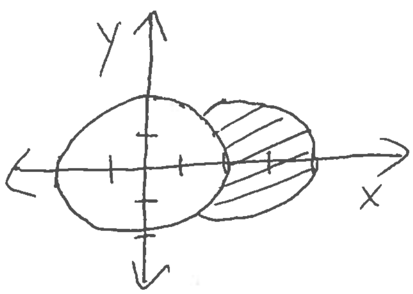
$$= -\frac{1}{2}e^{-x^2}\Big|_0^2 = -\frac{1}{2}e^{-4} - \left(-\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(1-e^{-4}\right)$$

Answer 1(c): $\frac{1}{2}\left(1-e^{-4}\right)$

2. (15 points) Calculate the area of the region inside the circle $r=4\cos\theta$ and outside the circle $r=2$.

Hint: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$.



$$\int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} (8\cos^2\theta - 2)\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} (4\cos 2\theta + 2)\,d\theta$$

$2 = 4\cos\theta$

$\frac{1}{2} = \cos\theta$

$\Rightarrow \theta = \pm\frac{\pi}{3}$

$$= 2\sin(2\theta) + 2\theta\Big|_{-\pi/3}^{\pi/3} = \left[2\left(\frac{\sqrt{3}}{2}\right) + \frac{2\pi}{3}\right] - \left[-2\left(\frac{\sqrt{3}}{2}\right) - \frac{2\pi}{3}\right]$$

$$= 2\sqrt{3} + \frac{4\pi}{3}$$

Answer 2: $\boxed{\frac{4\pi}{3} + 2\sqrt{3}}$

3. (15 points) If $R=((x,y):0\leqslant x\leqslant 6, 0\leqslant y\leqslant 6)$ and P is the partition of R into nine equal squares by the lines x = 2, x = 4, y = 2, and y = 4. Approximate $\iint_R f(x,y)\,dA$ by calculating the corresponding Riemann sum $\sum_{k=1}^{9} f(\bar{x}_k, \bar{y}_k)\ \Delta\ A_k$, assuming that $(\bar{x}_k, \bar{y}_k)$ are the centers of the nine squares. Take $f(x,y)=6+2x+3y$.

(1, 5)	(3, 5)	(5, 5)
(1, 3)	(3, 3)	(5, 3)
(1, 1)	(3, 1)	(5, 1)

$f(1,1)=11 \quad f(1,3)=17 \quad f(1,5)=23$

$f(3,1)=15 \quad f(3,3)=21 \quad f(3,5)=27$

$f(5,1)=19 \quad f(5,3)=25 \quad f(5,5)=31$

$A \approx 4(11+15+19+17+21+25+23+27+31)$

$= 4(45+63+81) = 4(189)$

$= 756$

Answer 3: $\boxed{756}$

4. (25 points) Calculate the surface area of the part of the saddle $az = x^2 - y^2$ inside the cylinder $x^2 + y^2 = a^2$, $a > 0$.

$$z = \frac{x^2}{a} - \frac{y^2}{a}$$

$$\frac{\partial z}{\partial x} = \frac{2x}{a} \qquad \frac{\partial z}{\partial y} = -\frac{2y}{a}$$

$$SA = \iint_R \sqrt{1 + \left(\frac{2x}{a}\right)^2 + \left(-\frac{2y}{a}\right)^2}\, dA = \iint_R \sqrt{1 + \frac{4x^2}{a^2} + \frac{4y^2}{a^2}}\, dA$$

$$= \int_0^{2\pi} \int_0^a \sqrt{1 + \frac{4r^2}{a^2}}\, r\,dr\,d\theta \qquad u = 1 + 4\frac{r^2}{a^2} \quad du = \frac{8r}{a^2}\,dr$$

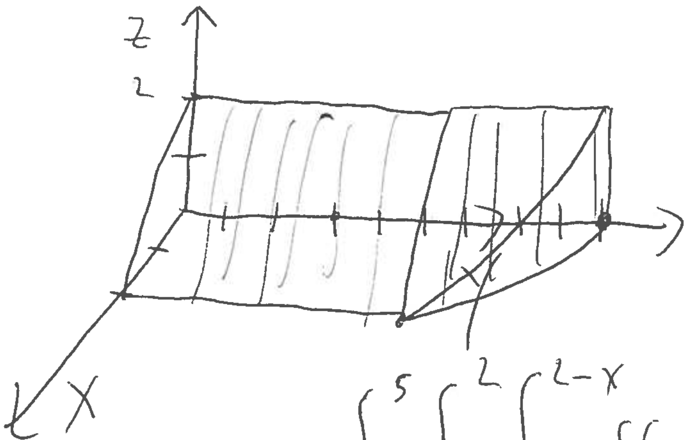
$$= \frac{a^2}{8} \int_0^{2\pi} \int_1^5 \sqrt{u}\, du\, d\theta$$

$$= \frac{a^2}{8} \int_0^{2\pi} \frac{2}{3} u^{3/2} \Big|_1^5 d\theta$$

$$= \int_0^{2\pi} \frac{a^2}{12} (5\sqrt{5} - 1)\, d\theta = \frac{\pi a^2 (5\sqrt{5} - 1)}{6}$$

Answer 4: $\frac{\pi a^2 (5\sqrt{5} - 1)}{6}$

5. (20 points) Rewrite the integral $\int_0^2 \int_0^{9-x^2} \int_0^{2-x} f(x,y,z)\,dz\,dy\,dx$ changing the order of integration to $dz\,dx\,dy$.



$y = 9 - x^2$

$x = \sqrt{9-y}$

$$\int_0^5 \int_0^2 \int_0^{2-x} f(x,y,z)\,dz\,dx\,dy$$

$$+ \int_5^9 \int_0^{\sqrt{9-y}} \int_0^{2-x} f(x,y,z)\,dz\,dx\,dy$$

Answer 5: $\int_0^5 \int_0^2 \int_0^{2-x} f(x,y,z)\,dz\,dx\,dy + \int_5^9 \int_0^{\sqrt{9-y}} \int_0^{2-x} f(x,y,z)\,dz\,dx\,dy$

6. (25 points) Evaluate the integral $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}}(x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx$.

Hint – Convert to spherical coordinates.

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{3}(\rho^2)^{3/2}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{3}\rho^5\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\frac{\rho^6}{6}\sin\phi\,\Big|_0^3\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\frac{243}{2}\sin\phi\,d\theta\,d\phi$$

$$= 243\pi\int_0^{\pi}\sin\phi\,d\phi = 243\pi(-\cos\phi)\Big|_0^{\pi}$$

$$= 243\pi\big[(-(-1))-(-1)\big] = 486\pi$$

Answer 6: $\boxed{486\pi}$

7. Find the divergence and curl of the following vector fields. (15 points each)

a) $\boldsymbol{F}(x, y, z) = x^2\boldsymbol{i} + y^2\boldsymbol{j} + z^2\boldsymbol{k}$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z$$

$$\nabla \times \vec{F} = \langle 0-0, 0-0, 0-0 \rangle$$

Divergence 7a): $2x + 2y + 2z$

Curl 7a): $\vec{0}$

b) $\boldsymbol{F}(x, y, z) = yz\,\boldsymbol{i} + xz\,\boldsymbol{j} + xy\,\boldsymbol{k}$

$$\nabla \cdot \vec{F} = 0 + 0 + 0 = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x-x)\hat{i} + (y-y)\hat{j} + (z-z)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

Divergence 7b): 0

Curl 7b): $\langle 0, 0, 0 \rangle$

8. (25 points) Prove that if the curl of a vector field is not zero, then the vector field is not conservative.

A vector field is conservative if it is the gradient of a function. So, if $\vec{F}$ is conservative we have

$$\vec{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

In this case

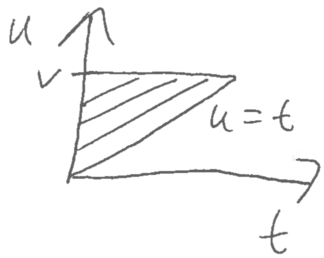
$$\nabla \times \vec{F} = \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle$$

From equality of mixed partials we get $\nabla \times \vec{F} = \vec{0}$.

So, if $\nabla \times \vec{F} \neq 0$, then $\vec{F} \neq \nabla f$, and $\vec{F}$ is not conservative Q.E.D.

Extra Credit (20 points): Prove the identity:

$$\int_0^x \int_0^v \int_0^u f(t)\,dt\,du\,dv = (1/2)\int_0^x (x-t)^2 f(t)\,dt$$



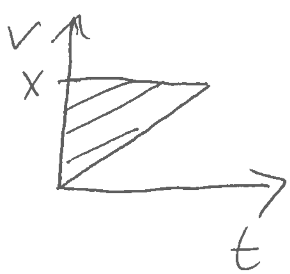
Switching the limits of integration:

$$= \int_0^x \int_0^v \int_t^v f(t)\,du\,dt\,dv$$

$$= \int_0^x \int_0^v u f(t) \Big|_{u=t}^{u=v} dt\,dv$$

$$= \int_0^x \int_0^v (v-t) f(t)\,dt\,dv$$

Switching the limits of integration again:



$$= \int_0^x \int_t^x (v-t) f(t)\,dv\,dt$$

$$= \int_0^x \left(\frac{v^2}{2} - tv\right) f(t) \Big|_{v=t}^{v=x} dt$$

$$= \int_0^x \left(\frac{x^2}{2} - tx\right) f(t)\,dt - \int_0^x \left(\frac{t^2}{2} - t^2\right) f(t)\,dt$$

$$= \int_0^x \left(\frac{x^2}{2} - tx + \frac{t^2}{2}\right) f(t)\,dt$$

$$= \boxed{\int_0^x \frac{1}{2}(x-t)^2 f(t)\,dt}$$

Q.E.D