Math2210 Midterm 2 Summer, 2012 Dylan Zwick

Name Key Date 7-19-2012

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. (20 pts) Find the directional derivative of $f(x,y)=x^2-3xy+2y^2$ at $\boldsymbol{p}=(-1,2)$ in the direction of $\boldsymbol{a}=2\boldsymbol{i}-\boldsymbol{j}$.

$$\nabla f = \langle 2x-3y, 4y-3x \rangle$$

$$\nabla f(-1,2) = \langle -8, 11 \rangle$$

$$\|a\| = \sqrt{2^2+(-1)^2} = \sqrt{5}$$

$$\hat{a} = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

$$D_{\hat{a}} f(-1,2) = \langle -8, 11 \rangle \cdot \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

$$= -\frac{16}{\sqrt{5}} - \frac{11}{\sqrt{5}} = -\frac{27}{\sqrt{5}}$$

Answer: $\boxed{-\frac{27}{\sqrt{5}}}$

2. (25 pts) For the surface $x^3y^2 - yz^2 + x^2y^3z - 5x + 2 = 7$ find the equation of the tangent plane at the point $(1,1,1)$.

$f_x = 3x^2y^2 + 2xy^3z - 5 \qquad f_x(1,1,1) = 3+2-5 = 0$

$f_y = 2x^3y - z^2 + 3x^2y^2z \qquad f_y(1,1,1) = 2-1+3 = 4$

$f_z = -2yz + x^2y^3 \qquad f_z(1,1,1) = -2+1 = -1$

$\nabla F(1,1,1) = \langle 0, 4, -1 \rangle$

Tangent Plane

$\nabla F \cdot \langle x-1, y-1, z-1 \rangle = 0$

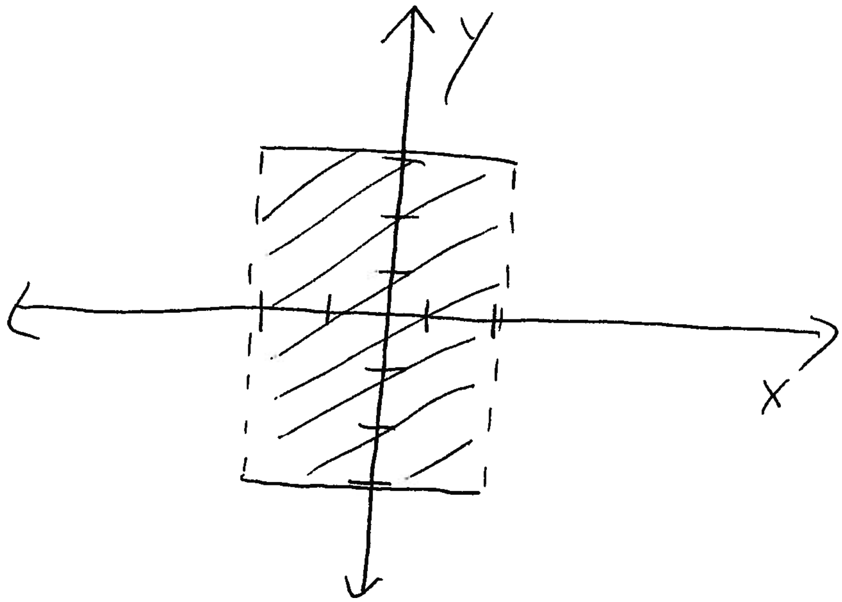
$\Rightarrow \langle 0, 4, -1 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0$

$4y - 4 - z + 1 = 0$

$4y - z = 3$

Tangent Plane: $\boxed{4y - z = 3}$

3. (8 pts) Draw the set $S=\{(x,y)|x\in(-2,2)\,y\in[-3,3]\}$.



(5 points) Is the set open, closed, or neither? Neither

(3 points) Is the point (0,-3) an interior point? No

(3 points) Is the point (2,-3) an interior point? No

(3 points) Is the point (0,-3) a boundary point? Yes

(3 points) Is the point (2,-3) a boundary point? Yes

4. (20 pts) Find all critical points for $f(x,y)=x^3+y^3-6xy$. Determine whether each point is a minimum, maximum or saddle point.

$$\nabla f = \langle 3x^2-6y,\ 3y^2-6x \rangle$$

$3x^2-6y=0$ $\quad y=\frac{1}{2}x^2$

$3y^2-6x=0$

$3\left(\frac{x^2}{2}\right)^2-6x=0 \quad \frac{3}{4}x^4-6x=0$

$(0,0)$ is a critical point. If $x \neq 0$

$\frac{3}{4}x^3=6 \Rightarrow x^3=8 \quad x=2,\ y=2$

$(2,2)$ is a critical point.

$f_{xx}=6x \qquad D=(6x)(6y)-(-6)^2$

$f_{yy}=6y \qquad \quad =36xy-36$

$f_{xy}=-6 \qquad D(0,0)=-36 \Rightarrow$ Saddle Point

$D(2,2)=108 \quad f_{xx}(2,2)=12 \Rightarrow$ Minimum.

Critical point(s) (Specify whether they're min, max or saddle.):

$(0,0)$, Saddle $\quad (2,2)$, Minimum

5. For $z=f(x,y)=x^2+y^2+e^{x+y^2}$, find

(a) (10 pts) $\dfrac{\partial z}{\partial y}$ at (1, 1)

$$\frac{\partial z}{\partial y} = 2y + 2ye^{x+y^2}$$

$$\frac{\partial z}{\partial y}(1,1) = 2 + 2e^2$$

Answer : $2+2e^2$

(b) (10 pts) f_{xy}

$$f_{xy} = f_{yx} = 2ye^{x+y^2}$$

Answer : $2ye^{x+y^2}$

6. (20 points) Find the minimum of $x^2+4xy+y^2$ subject to the constraint $x-y-6=0$.

$f(x,y) = x^2+4xy+y^2$

$g(x,y) = x-y-6$

$\nabla f = \langle 2x+4y, 4x+2y \rangle$

$\nabla g = \langle 1, -1 \rangle$

$\nabla f = \lambda \nabla g \Rightarrow$ $2x+4y = \lambda$
$4x+2y = -\lambda$
$x-y = 6$

$4x+2y = -2x-4y$

$6x+6y = 0 \Rightarrow x+y=0$
$x-y = 6$

$\Rightarrow 2x = 6 \Rightarrow x = 3$
$y = -3$

$f(3,-3) = 3^2+4(3)(-3)+(-3)^2$
$= 9-36+9 = -18$

Answer: $\boxed{-18}$

7. (25 points) Find the dimensions of the rectangular box of volume V_0 for which the sum of the edge lengths is least.

Edge lengths x, y, z.

$x+y+z \qquad V = xyz = V_0$

$f(x,y) = x + y + \frac{V_0}{xy}$

$\nabla f = \left\langle 1 - \frac{V_0}{x^2 y}, 1 - \frac{V_0}{xy^2} \right\rangle$

$V_0 = x^2 y, \quad V_0 = xy^2$

$y = \frac{V_0}{x^2} \qquad V_0 = x\left(\frac{V_0^2}{x^4}\right) \Rightarrow x^3 = V_0 \Rightarrow x = V_0^{1/3}$

$y = V_0^{1/3}$

$f_{xx} = \frac{2V_0}{x^3 y} \quad f_{yy} = \frac{2V_0}{xy^3} \quad f_{xy} = \frac{V_0}{x^2 y^2}$

$D = \frac{4V_0^2}{x^4 y^4} - \left(\frac{V_0}{x^2 y^2}\right)^2 = \frac{3V_0^2}{x^4 y^4}$

$D(V_0^{1/3}, V_0^{1/3}) = \frac{3}{V_0^{2/3}} > 0$

$f_{xy}(V_0^{1/3}, V_0^{1/3}) = \frac{2}{V_0^{1/3}} > 0 \quad \Rightarrow$ Min.

$x = V_0^{1/3} \quad y = V_0^{1/3} \quad z = V_0^{1/3}$

$z = \frac{V_0}{V_0^{2/3}} = V_0^{1/3}$

Answer: All sides length $V_0^{1/3}$

8. (20 pts) Find $\frac{dw}{dt}$ if $w(x,y)=x^2y^3$, $x=t^3$, $y=t^2$. (Your answer must be only in terms of t.)

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial w}{\partial x} = 2xy^3 \qquad \frac{\partial w}{\partial y} = 3x^2y^2$$

$$\frac{dx}{dt} = 3t^2 \qquad \frac{dy}{dt} = 2t$$

$$\frac{dw}{dt} = (2xy^3)(3t^2) + (3x^2y^2)(2t)$$

$$= 6(t^3)(t^2)^3t^2 + 3(t^3)^2(t^2)^2(2t)$$

$$= 6t^{11} + 6t^{11} = 12t^{11}$$

Answer : $\boxed{12t^{11}}$

9. Find the limit, if it exists. (Show all your reasoning.)

(a) $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^3-3x^2+3xy^2-y^3}{y-2x^2}$ (10 pts)

It's a rational function, and $y-2x^2 \neq 0$ at $(1,1)$, so it's continuous. Thus, the limit is

$$\frac{1-3+3-1}{1-2} = \frac{0}{-1} = 0$$

Answer : $\boxed{0}$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\tan(x^2+y^2)}{\sqrt{x^2+y^2}}$ (15 pts)

Convert to polar

$$\lim_{r\to 0} \frac{\tan(r^2)}{r}$$ L'Hospital

$$\Rightarrow \lim_{r\to 0} \frac{2r\sec^2(r^2)}{1} = 2(0)(1^2) = 0$$

Answer : $\boxed{0}$