Math2210 Midterm 1 Summer, 2012 Dylan Zwick

Name: Key Date: 7-12-2012

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. For $x=2\sqrt{t-2}$ and $y=3\sqrt{4-t}$ such that $2 \le t \le 4$, do the following:

(a) (10 pts) Eliminate the parameter to obtain the corresponding Cartesian equation.

$x = 2\sqrt{t-2}$ $\quad$ $y = 3\sqrt{4-t}$

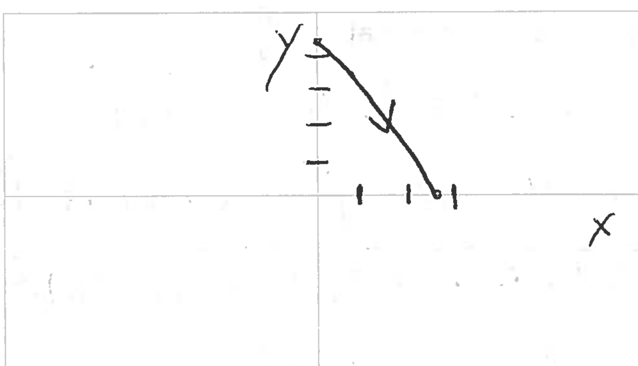
$x^2 = 4(t-2)$ $\quad$ $y^2 = 9(4-t)$

$\frac{x^2}{4} + 2 = t$ $\quad$ $\frac{y^2}{9} = 4 - t \Rightarrow t = 4 - \frac{y^2}{9}$ $\quad$ $\frac{x^2}{4} + 2 = 4 - \frac{y^2}{9}$

$\frac{x^2}{4} + \frac{y^2}{9} = 2$

Answer 1(a): $\boxed{\frac{x^2}{4} + \frac{y^2}{9} = 2}$

(b) (10 pts) Graph the curve.



(c) (5 pts) Indicate if the curve is simple and/or closed.

Simple: T or F (circle one)

Closed: T or F (circle one)

2. (10 pts) Find the length of the curve given by $x=4\sqrt{t}$ and $y=t^2+\frac{1}{2t}$ for $\frac{1}{4}\le t\le 1$.

$$\frac{dx}{dt} = \frac{2}{\sqrt{t}} \qquad \frac{dy}{dt} = 2t - \frac{1}{2t^2}$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{4}{t} \qquad \left(\frac{dy}{dt}\right)^2 = 4t^2 - \frac{2}{t} + \frac{1}{4t^4}$$

$$L = \int_{1/4}^{1} \sqrt{\frac{4}{t} + 4t^2 - \frac{2}{t} + \frac{1}{4t^4}}\, dt = \int_{1/4}^{1} \sqrt{4t^2 + \frac{2}{t} + \frac{1}{4t^4}}\, dt$$

$$= \int_{1/4}^{1} \sqrt{\left(2t + \frac{1}{2t^2}\right)^2}\, dt = \int_{1/4}^{1} \left(2t + \frac{1}{2t^2}\right) dt$$

$$= t^2 - \frac{1}{2t}\Big|_{1/4}^{1} = \left(1 - \frac{1}{2}\right) - \left(\frac{1}{16} - 2\right) = 3 - \frac{1}{2} - \frac{1}{16} = \frac{48}{16} - \frac{8}{16} - \frac{1}{16}$$

Answer 2: $= \boxed{\frac{39}{16}}$

3. (15 pts) For position vector given by $\boldsymbol{r}(t)=\sin 2t\,\boldsymbol{i}+\cos 3t\,\boldsymbol{j}+\cos 4t\,\boldsymbol{k}$, find the velocity and acceleration vectors and the speed at $t=\frac{\pi}{4}$.

$$\vec{v}(t) = 2\cos(2t)\,\hat{i} - 3\sin(3t)\,\hat{j} - 4\sin(4t)\,\hat{k}$$

$$\vec{a}(t) = -4\sin(2t)\,\hat{i} - 9\cos(3t)\,\hat{j} - 16\cos(4t)\,\hat{k}$$

$$\cancel{\vec{v}\left(\frac{\pi}{4}\right) = 0\,\hat{i} + \frac{3}{\sqrt{2}}\,\hat{j} + \hat{k}} \qquad \cancel{\left\|\vec{v}\left(\frac{\pi}{4}\right)\right\| = \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2} = \frac{3\sqrt{2}}{2}}$$

$\boldsymbol{v}(t)$ = $2\cos(2t)\,\hat{i} - 3\sin(3t)\,\hat{j} - 4\sin(4t)\,\hat{k}$

$\boldsymbol{a}(t)$ = $-4\sin(2t)\,\hat{i} - 9\cos(3t)\,\hat{j} - 16\cos(4t)\,\hat{k}$

$$\vec{v}\left(\frac{\pi}{4}\right) = 0\,\hat{i} + \frac{3}{\sqrt{2}}\,\hat{j} + 0\,\hat{k} \qquad \left\|\vec{v}\left(\frac{\pi}{4}\right)\right\| = \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2} = \frac{3\sqrt{2}}{2}$$

Speed at $\frac{\pi}{4}$ = $\boxed{\frac{3\sqrt{2}}{2}}$

4. (10 pts) Find the limit, if it exists. $\lim_{t\to 0^+}\left[\ln(t^3)\boldsymbol{i}+t^2\ln(t)\boldsymbol{j}+t\boldsymbol{k}\right]$

$$\lim_{t\to 0^+} \ln(t^3) = -\infty$$

$$\lim_{t\to 0^+} t^2\ln(t) = \lim_{t\to 0^+} \frac{\ln(t)}{\frac{1}{t^2}} = \frac{\lim_{t\to 0^+} \frac{1}{t}}{\lim_{t\to 0^+} -\frac{2}{t^3}}$$

$$= \lim_{t\to 0^+} -\frac{t^2}{2} = 0$$

$$\lim_{t\to 0^+} t = 0$$

Answer (4) : $-\infty\hat{i} + 0\hat{j} + 0\hat{k}$ Does not Exist

5. (10 pts) Find the equation of the sphere that has the line segment joining (3, 1, 7) and (7, 5, 5) as a diameter.

$$\text{Midpoint} = \left(\frac{3+7}{2}, \frac{1+5}{2}, \frac{7+5}{2}\right) = (5, 3, 6)$$

$$r = \sqrt{(5-3)^2+(3-1)^2+(6-7)^2}$$
$$= \sqrt{4+4+1}$$
$$= \sqrt{9} = 3$$

Radius = 3

center = (5, 3, 6)

Eqn of sphere: $(x-5)^2+(y-3)^2+(z-6)^2 = 9$

6. (10 pts each) Let $\boldsymbol{a}=\langle 4,1,2\rangle$, $\boldsymbol{b}=\langle 2,4,1\rangle$ and $\boldsymbol{c}=6\boldsymbol{i}+3\boldsymbol{j}$. Find each of the following.

(a) $2\boldsymbol{a}-3\boldsymbol{c}$

$$\langle 8,2,4\rangle-\langle 18,9,0\rangle$$
$$=\langle -10,-7,4\rangle$$

$2\boldsymbol{a}-3\boldsymbol{c}$ = $-10\hat{i}-7\hat{j}+4\hat{k}$

(b) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$

$$\vec{b}+\vec{c}=\langle 2,4,1\rangle+\langle 6,3,0\rangle$$
$$=\langle 8,7,1\rangle$$
$$\vec{a}\cdot(\vec{b}+\vec{c})=\langle 4,1,2\rangle\cdot\langle 8,7,1\rangle$$
$$=32+7+2=41$$

$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$ = 41

(c) $\boldsymbol{b}\cdot\boldsymbol{c}-|\boldsymbol{b}|$

$$\vec{b}\cdot\vec{c}=\langle 2,4,1\rangle\cdot\langle 6,3,0\rangle=12+12=24$$
$$\|\vec{b}\|=\sqrt{2^2+4^2+1^2}=\sqrt{4+16+1}=\sqrt{21}$$

$\boldsymbol{b}\cdot\boldsymbol{c}-|\boldsymbol{b}|$ = $24-\sqrt{21}$

(Note: This is # 6 continued $a=\langle 4,1,2\rangle$, $b=\langle 2,4,1\rangle$ and $c=6i+3j$

(d) $\hat{c}$ (the unit vector)

$$\|\vec{c}\| = \sqrt{6^2+3^2} = \sqrt{36+9} = \sqrt{45}$$

$$= 3\sqrt{5}$$

$$\hat{c} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle$$

$\hat{c}$ = $\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle$

(e) $a\times(b\times c)$

$$\vec{b}\times\vec{c} = \langle 2,4,1\rangle \times \langle 6,3,0\rangle$$

$$= \langle 4(0)-1(3),\ 1(6)-2(0),\ 2(3)-4(6)\rangle$$

$$= \langle -3, 6, -18\rangle$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \langle 4,1,2\rangle \times \langle -3, 6, -18\rangle$$

$$= \langle -18-18, -6+72, 24+3\rangle = \langle -36, 66, 27\rangle$$

$a\times(b\times c)$ = $\langle -36, 66, 27\rangle$

(f) $a\cdot(b\times c)$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \langle 4,1,2\rangle \cdot \langle -3, 6, -18\rangle$$

$$= -12+6-36 =$$

$a\cdot(b\times c)$ = -42

7. (10 pts each) For $\boldsymbol{a}=3\boldsymbol{i}+4\boldsymbol{j}+5\boldsymbol{k}$ and $\boldsymbol{b}=2\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}$, find each of the following:

(a) Direction cosines for $\boldsymbol{a}$.

$$\|\vec{a}\| = \sqrt{3^2+4^2+5^2} = \sqrt{50} = 5\sqrt{2}$$

$\cos\alpha = \dfrac{3}{5\sqrt{2}}$

$\cos\beta = \dfrac{4}{5\sqrt{2}}$

$\cos\gamma = \dfrac{1}{\sqrt{2}}$

(b) The angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$. (Just write a simplified expression. If you don't have a calculator just write the numerical formula for the angle.)

$$\vec{a}\cdot\vec{b} = \langle 3,4,5\rangle \cdot \langle 2,1,3\rangle = 6+4+15 = 25$$

$$\|\vec{a}\| = 5\sqrt{2}$$

$$\|\vec{b}\| = \sqrt{2^2+1^2+3^2} = \sqrt{14}$$

$$\theta = \cos^{-1}\left(\frac{25}{10\sqrt{7}}\right) = \cos^{-1}\left(\frac{5}{2\sqrt{7}}\right)$$

$\theta = \cos^{-1}\left(\dfrac{5}{2\sqrt{7}}\right)$

(c) Find the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$.

$$\text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a} = \left(\frac{25}{50}\right)\langle 3,4,5\rangle = \left\langle \frac{3}{2}, 2, \frac{5}{2}\right\rangle$$

Projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ = $\left\langle \frac{3}{2}, 2, \frac{5}{2}\right\rangle$

8. (10 pts each) For the planes given by

$$3x - 2y + 5z = 16$$

and

$$4x + 2y + z = 13,$$

answer the following questions.

(a) Find the line of intersection between the planes and write that line in parametric equations.

$\vec{n}_1 = \langle 3, -2, 5 \rangle$

$\vec{n}_2 = \langle 4, 2, 1 \rangle$

$\vec{n}_1 \times \vec{n}_2 = \langle -12, 17, 14 \rangle$

$x = 0$

$-2y + 5z = 16$

$2y + z = 13$

$6z = 29$

$z = \frac{29}{6}$

$2y = \frac{78}{6} - \frac{29}{6} = \frac{49}{6}$

$y = \frac{49}{12}$

$\left(0, \frac{49}{12}, \frac{29}{6}\right)$

Line: $x(t) = 0 - 12t \quad y(t) = \frac{49}{12} + 17t \quad z(t) = \frac{29}{6} + 14t$

(b) Find the equation of the plane that is perpendicular to the line of intersection and goes through the point (1, 3, 2).

$-12(1) + 17(3) + 14(2) = -12 + 51 + 28 = 67$

Equation of plane: $-12x + 17y + 14z = 67$

9. (a) (10 pts) Convert $x^2+y^2=2y+2x$ ~~4z~~ from a Cartesian coordinate equation into an equation in cylindrical coordinates.

$x^2+y^2=r^2 \quad x=r\cos\theta \quad y=r\sin\theta$

$r^2=2r\sin\theta+2r\cos\theta$ ~~4z~~

$r=2(\sin\theta+\cos\theta)$

Answer : $r=2(\sin\theta+\cos\theta)$

(b) (10 pts) Convert $r=2\sin\theta$ from a cylindrical coordinate equation into an equation in Cartesian coordinates.

$r=\sqrt{x^2+y^2} \Rightarrow r=2\sin\theta$

$\sin\theta=\frac{y}{\sqrt{x^2+y^2}} \quad r=2(\cancel{y}/r)$

$\Rightarrow r^2=2y$

$x^2+y^2=2y$

$x^2+(y-1)^2=1$

Answer : $x^2+(y-1)^2=1$