

Name Key Date 8-1-2012

**Instructions:** Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. (15 points) For position vector given by  $\mathbf{r}(t) = (t^3 - 2t^2 + 5t)\mathbf{i} + (t^2 + t + 1)\mathbf{j}$ , find the velocity and acceleration vectors and the speed at  $t=1$ .

$$\vec{v}(t) = (3t^2 - 4t + 5)\hat{i} + (2t + 1)\hat{j}$$

$$v(t) \text{ (6 points)} = (3t^2 - 4t + 5)\hat{i} + (2t + 1)\hat{j}$$

$$\vec{a}(t) = (6t - 4)\hat{i} + 2\hat{j}$$

$$a(t) \text{ (6 points)} = (6t - 4)\hat{i} + 2\hat{j}$$

$$\vec{v}(1) = (3 - 4 + 5)\hat{i} + (2 + 1)\hat{j} = 4\hat{i} + 3\hat{j}$$

$$\|\vec{v}(1)\| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

$$\text{speed at } t=1 \text{ (3 points)} = 5$$

2. (15 points) Let  $\mathbf{a} = \langle 2, -1, 3 \rangle$ ,  $\mathbf{b} = \langle 1, 3, 2 \rangle$  and  $\mathbf{c} = \langle 0, 1, -1 \rangle$ . Find each of the following.

(a)  $2\mathbf{a} - 3\mathbf{c}$  (3 points)

$$2\vec{a} = \langle 4, -2, 6 \rangle$$

$$3\vec{c} = \langle 0, 3, -3 \rangle$$

$$2\vec{a} - 3\vec{c} = \langle 4, -2, 6 \rangle - \langle 0, 3, -3 \rangle$$

$$= \langle 4, -5, 9 \rangle$$

$$2\mathbf{a} - 3\mathbf{c} = \underline{\langle 4, -5, 9 \rangle}$$

(b)  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$  (3 points)

$$\vec{b} + \vec{c} = \langle 1, 4, 1 \rangle$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \langle 2, -1, 3 \rangle \cdot \langle 1, 4, 1 \rangle$$

$$= 2 - 4 + 3 = 1$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \underline{1}$$

(c) projection of  $\mathbf{a}$  onto  $\mathbf{b}$  (6 points)

$$\text{proj}_{\mathbf{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b} = \frac{5}{14} \langle 1, 3, 2 \rangle$$

$$\vec{a} \cdot \vec{b} = 2 - 3 + 6 = 5$$

$$\vec{b} \cdot \vec{b} = 1 + 9 + 4 = 14$$

$$\text{projection of } \mathbf{a} \text{ onto } \mathbf{b} = \underline{\langle \frac{5}{14}, \frac{15}{14}, \frac{10}{14} \rangle}$$

(d)  $\hat{\mathbf{a}}$  (the unit vector) (3 points)

$$\hat{\mathbf{a}} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\sqrt{14}} \langle 2, -1, 3 \rangle$$

$$\begin{aligned} \|\vec{a}\| &= \sqrt{2^2 + (-1)^2 + 3^2} \\ &= \sqrt{14} \end{aligned}$$

$$\hat{\mathbf{a}} = \underline{\frac{1}{\sqrt{14}} \langle 2, -1, 3 \rangle}$$

3. (20 points) For the points A(1, 3, 2), B(0, 3, 0) and C(2, 4, 3)

(a) (10 points) Find a normal vector to, and equation for, the plane through points A, B and C.

$$\vec{BA} = \langle 1, 0, 2 \rangle$$

$$\vec{BC} = \langle 2, 1, 3 \rangle$$

$$\vec{BA} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 2 & 1 & 3 \end{vmatrix} = -2\hat{i} + \hat{j} + \hat{k}$$

$$-2x + (y - 3) + z = 0$$

Normal vector =  $\langle -2, 1, 1 \rangle$

Equation of plane:  $-2x + y + z = 3$

(b) (10 points) Write a set of parametric equations for the line through point B and perpendicular to the plane in part (a).

$$\vec{r}(t) = \langle -2, 1, 1 \rangle t + \langle 0, 3, 0 \rangle$$

Line:  $x(t) = -2t \quad y(t) = t + 3 \quad z(t) = t$

4. (20 points) Find the directional derivative of  $f(x, y, z) = x^2y + y^2z + z^2x$  at  $p=(1, 0, 1)$  in the direction of  $a = i + 2j + 2k$ .

$$\frac{\partial f}{\partial x} = 2xy + z^2$$

$$\frac{\partial f}{\partial y} = x^2 + 2yz$$

$$\frac{\partial f}{\partial z} = y^2 + 2xz$$

$$\nabla f(1, 0, 1) = \langle 1, 1, 2 \rangle$$

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{\langle 1, 2, 2 \rangle}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\langle 1, 2, 2 \rangle}{3}$$

$$D_{\hat{a}} f(1, 0, 1) = \langle 1, 1, 2 \rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$$

$$= \frac{1}{3} + \frac{2}{3} + \frac{4}{3} = \frac{7}{3}$$

Answer: \_\_\_\_\_

$$\boxed{\frac{7}{3}}$$

5. (20 points) Find all critical points of the function  $f(x, y) = xy^2 - 6x^2 - 3y^2$ . Determine if each critical point is a local minimum, a local max, or neither. If there are none of the given type of point, just write "None".

Polynomials are always differentiable, and there is no boundary, so all critical points are stationary points.

$$\frac{\partial f}{\partial y} = 2xy - 6y \quad \frac{\partial f}{\partial x} = y^2 - 12x \quad \text{Want } \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow x = \frac{y^2}{12} \quad \frac{y^3}{6} - 6y = 0 \Rightarrow y = 0, \pm 6$$

$$x = 0, 3$$

So, critical points  $(0, 0), (3, 6), (3, -6)$

$$\frac{\partial^2 f}{\partial x^2} = -12 \quad \frac{\partial^2 f}{\partial y^2} = 2x - 6 \quad \frac{\partial^2 f}{\partial x \partial y} = 2y$$

$$D(x, y) = 72 - 24x - 4y^2 = f_{xx}f_{yy} - (f_{xy})^2$$

$$D(0, 0) = 72 > 0, \quad \frac{\partial^2 f}{\partial x^2} = -12 < 0 \Rightarrow \text{Max}$$

$$D(3, 6) = -144 < 0 \Rightarrow \text{saddle}$$

$$D(3, -6) = -144 < 0 \Rightarrow \text{saddle}$$

Critical Points:  $(0, 0), (3, 6), (3, -6)$

Local Max:  $(0, 0)$

Local Min: None

Neither:  $(3, 6), (3, -6)$

Saddle

Indeterminate: None

6. (20 points) Calculate the integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

Note - You must provide a correct calculation. Don't just state the answer.

$$\begin{aligned} \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \quad \text{Convert to polar} \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \quad u = r^2 \ du = 2rdr \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^{\infty} e^{-u} du d\theta = \frac{1}{2} \int_0^{2\pi} -e^{-u} \Big|_0^{\infty} d\theta \\ &= \frac{1}{2} \int_0^{2\pi} d\theta = \frac{2\pi}{2} = \pi \end{aligned}$$

$$\Rightarrow \int_0^{\infty} e^{-x^2} dx = \sqrt{\cancel{\pi}} \sqrt{\pi}$$

Answer:  $\sqrt{\pi}$

7. (20 points) Calculate the surface area of the part of the elliptic paraboloid  
 $z=16-x^2-y^2$  above the xy-plane.

$$\frac{\partial z}{\partial x} = -2x \quad \frac{\partial z}{\partial y} = -2y \quad \iint_R \sqrt{1+(2x)^2+(2y)^2} dA$$

$$\int_0^{2\pi} \int_0^4 \sqrt{1+4r^2} r dr d\theta \quad = \iint_R \sqrt{1+4x^2+4y^2} dA$$

$$u = 1+4r^2$$

$$du = 8rdr$$

$$\Rightarrow \frac{1}{8} \int_0^{2\pi} \int_1^{65} \sqrt{u} du d\theta = \int_0^{2\pi} \frac{u^{3/2}}{12} \Big|_1^{65} d\theta$$

$$= \int_0^{2\pi} \left( \frac{65\sqrt{65}-1}{12} \right) d\theta = 2\pi \left( \frac{65\sqrt{65}-1}{12} \right)$$

$$= \pi \left( \frac{65\sqrt{65}-1}{6} \right)$$

Surface Area:  $\pi \left( \frac{65\sqrt{65}-1}{6} \right)$

8. (30 points) Calculate the following integrals:

a)  $\int_0^{\frac{\pi}{2}} \int_0^{\sin y} e^x \cos y dx dy$  (10 points)

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \int_0^{\sin y} e^x \cos y dx dy = \int_0^{\frac{\pi}{2}} e^x \cos y \Big|_0^{\sin y} dy \\ &= \int_0^{\frac{\pi}{2}} (e^{\sin y} - 1) \cos y dy = \int_0^{\frac{\pi}{2}} e^{\sin y} \cos y dy - \int_0^{\frac{\pi}{2}} \cos y dy \\ &= e^{\sin y} - \sin y \Big|_0^{\frac{\pi}{2}} = (e - 1) - (1 - 0) = e - 2 \end{aligned}$$

Answer:  $e - 2$

b)  $\int_0^{\frac{\pi}{2}} \int_0^z \int_0^y \sin(x+y+z) dx dy dz$  (10 points)

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \int_0^z -\cos(x+y+z) \Big|_{x=0}^{x=y} dy dz \\ &= \int_0^{\frac{\pi}{2}} \int_0^z [\cos(y+z) - \cos(2y+z)] dy dz \\ &= \int_0^{\frac{\pi}{2}} [\sin(y+z) - \frac{\sin(2y+z)}{2} \Big|_0^z] dz \\ &= \int_0^{\frac{\pi}{2}} \left[ (\sin(2z) - \frac{\sin(3z)}{2}) - (\sin(z) - \frac{\sin(z)}{2}) \right] dz \\ &= \int_0^{\frac{\pi}{2}} \left( \sin(2z) - \frac{\sin(3z)}{2} - \frac{\sin(z)}{2} \right) dz \\ &= \frac{\cos(3z)}{6} + \frac{\cos(2z)}{2} - \frac{\cos(z)}{2} \Big|_0^{\frac{\pi}{2}} \\ &= (0 + 0 - (-\frac{1}{2})) - \left(\frac{1}{6} + \frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \end{aligned}$$

Answer:  $\frac{1}{3}$

c)  $\int_0^2 \int_x^2 6x e^{y^3} dy dx$  (10 points)



Switching the order of integration:

$$\int_0^2 \int_0^y 6x e^{y^3} dx dy$$

$$= \int_0^2 3y^2 e^{y^3} dy = e^{y^3} \Big|_0^2 = e^8 - 1$$

Answer:

$e^8 - 1$

9. (20 points) Given  $\mathbf{F}(x, y, z) = 5x^3yz\mathbf{i} - 2yx^2\mathbf{j} + y^3z^2\mathbf{k}$ , calculate the following.

(a)  $\operatorname{div} \mathbf{F}$  (5 points)

$$15x^2yz - 2x^2 + 2y^3z$$

$$\operatorname{div} \mathbf{F} = \underline{15x^2yz - 2x^2 + 2y^3z}$$

$$\mathbf{F}(x, y, z) = 5x^3yz\hat{i} - 2yx^2\hat{j} + y^3z^2\hat{k}$$

(b) curl  $\mathbf{F}$  (5 points)

$$\left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5x^3yz & -2yx^2 & y^3z^2 \end{array} \right| = (3y^2z^2)\hat{i} + (5x^3y - 0)\hat{j} + (-4xy - 5x^3z)\hat{k}$$

(c)  $\nabla(\nabla \cdot \mathbf{F})$  (5 points)

$$\text{curl } \mathbf{F} = \langle 3y^2z^2, 5x^3y, -4xy - 5x^3z \rangle$$

$$\begin{aligned} & \nabla(15x^2yz - 2x^2 + 2y^3z) \\ &= \langle 30xyz - 4x, 15x^2z + 6y^2z, 15x^2y + 2y^3 \rangle \end{aligned}$$

(d)  $\nabla \cdot (\nabla \times \mathbf{F})$  (5 points)

$$\nabla(\nabla \cdot \mathbf{F}) = \langle 30xyz - 4x, 15x^2z + 6y^2z, 15x^2y + 2y^3 \rangle$$

$$= 0 + 5x^3 - 5x^3 = 0$$

Note: In general,  $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ .

$$\nabla \cdot (\nabla \times \mathbf{F}) = \boxed{0}$$

10. (30 points)

a) (20 points) Determine if the field  $\mathbf{F} = (2xy + z^2)\mathbf{i} + x^2\mathbf{j} + (2xz + \pi \cos \pi z)\mathbf{k}$  is conservative. If it is conservative, find a function  $f$  such that  $\nabla(f) = \mathbf{F}$ .

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^2 & x^2 & 2xz + \pi \cos \pi z \end{vmatrix}$$

$$= 0\hat{i} + (2z - 2z)\hat{j} + (2x - 2x)\hat{k} = \vec{0}$$

So, conservative.

$$\frac{\partial f}{\partial x} = 2xy + z^2 \Rightarrow f = x^2y + xz^2 + C(y, z)$$

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial(C(y, z))}{\partial y} = x^2 \Rightarrow \frac{\partial(C(y, z))}{\partial y} = 0 \Rightarrow C(y, z) = C(z)$$

$$\frac{\partial f}{\partial z} = 2xz + C'(z) = 2xz + \pi \cos(\pi z)$$

$$\Rightarrow C(z) = \sin(\pi z) + C$$

$$f = x^2y + xz^2 + \sin(\pi z) + C$$

Conservative? (Circle One)

True

False

$$f \text{ (if it exists)} = \underline{x^2y + xz^2 + \sin(\pi z) + C}$$

Note: Any particular choice of  $C$   
is also acceptable.

(This is problem 10 continued, so  $\mathbf{F} = (2xy + z^2)\mathbf{i} + x^2\mathbf{j} + (2xz + \pi \cos \pi z)\mathbf{k}$  ).

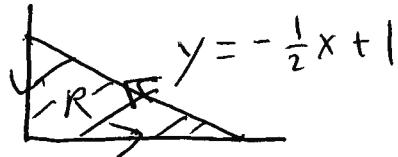
- b) (10 points) Calculate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where C is the line segment connecting the points  $(0,0,0)$  and  $(1,2,3)$ .

$$\begin{aligned}\int_{(0,0,0)}^{(1,2,3)} \mathbf{F} \cdot d\vec{r} &= f(1,2,3) - f(0,0,0) \\ &= (2+9+0) - (0+0+0) \\ &= 11\end{aligned}$$

Answer: \_\_\_\_\_ 

11. (20 points) Calculate the line integral  $\oint_C xy \, dx + (x+y) \, dy$  where C is the triangle with vertices (0,0), (2,0), and (0,1).

Hint – Use Green's theorem.



$$\frac{\partial N}{\partial x} = 1 \quad \frac{\partial M}{\partial y} = x$$

$$\oint_C xy \, dx + (x+y) \, dy = \iint_R (1-x) \, dA$$

$$= \int_0^2 \int_0^{-\frac{x}{2}+1} (1-x) \, dy \, dx = \int_0^2 (1-x)y \Big|_0^{1-\frac{x}{2}} \, dx$$

$$= \int_0^2 (1-x)(1-\frac{x}{2}) \, dx = \int_0^2 \left(1 - \frac{3}{2}x + \frac{x^3}{2}\right) \, dx$$

$$= x - \frac{3}{4}x^2 + \frac{x^3}{6} \Big|_0^2 = 2 - 3 + \frac{8}{6}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Answer: \_\_\_\_\_

$$\boxed{\frac{1}{3}}$$

12. (20 points) Calculate the surface integral  $\iint_G g(x, y, z) dS$  where  $g = x + y$  and  $G$  is the part of  $z = \sqrt{4 - x^2}$  with  $0 \leq x \leq \sqrt{3}$  and  $0 \leq y \leq 1$ .

Note -  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{4-x^2}} \quad \frac{\partial z}{\partial y} = 0$$

$$dS = \sqrt{1 + \frac{x^2}{4-y^2}} dA = \frac{z}{\sqrt{4-y^2}}$$

$$\Rightarrow 2 \iint_R \frac{x+y}{\sqrt{4-x^2}} dA = 2 \int_0^{\sqrt{3}} \int_0^1 \frac{x+y}{\sqrt{4-y^2}} dy dx$$

$$= 2 \int_0^{\sqrt{3}} \frac{xy + \frac{y^2}{2}}{\sqrt{4-x^2}} \Big|_0^1 dx$$

$$= 2 \int_0^{\sqrt{3}} \frac{x + \frac{1}{2}}{\sqrt{4-x^2}} dx = 2 \int_0^{\sqrt{3}} \frac{x}{\sqrt{4-x^2}} dx + \int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$$

$$= -2 \sqrt{4-x^2} + \sin^{-1}\left(\frac{x}{2}\right) \Big|_0^{\sqrt{3}}$$

$$= \left(-2 + \frac{\pi}{3}\right) - (-4+0) = 2 + \frac{\pi}{3}$$

Answer:  $2 + \frac{\pi}{3}$

**Extra Credit:**

(10 points) What did you think of the class, and what could be done to improve it?

Dylan is the best looking  
instructor ever!!!

Thank you all for a  
great class. Good luck  
in all you do.

-Dylan

A handwritten signature consisting of the letters 'D' and 'Y' stacked vertically.

134493 84/22

of parameterized problems. This is a new approach to solving hard combinatorial problems. It is based on the principle of "divide and conquer".

The basic idea is to divide a large problem into smaller subproblems, solve them independently, and then combine their solutions to solve the original problem. This is done by using a recursive algorithm that splits the problem into two or more subproblems, solves them, and then combines their results.

The algorithm starts with a large problem and divides it into smaller subproblems until they are small enough to be solved directly. Then, the results of these subproblems are combined to solve the original problem.

This approach has been used to solve many difficult problems in computer science, such as the Traveling Salesman Problem, the Knapsack Problem, and the Bin-Packing Problem.

The main advantage of this approach is that it can be applied to a wide range of problems, and it can be used to solve problems that are too large to be solved by other methods.

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