Math2210 Final Exam Summer, 2012 Dylan Zwick

Name Key Date 8-1-2012

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. *(15 points)* For position vector given by $\boldsymbol{r}(t)=(t^3-2t^2+5t)\boldsymbol{i}+(t^2+t+1)\boldsymbol{j}$, find the velocity and acceleration vectors and the speed at $t=1$.

$\vec{v}(t) = (3t^2-4t+5)\hat{i} + (2t+1)\hat{j}$

$\boldsymbol{v}(t)$ (6 points) = $(3t^2-4t+5)\hat{i}+(2t+1)\hat{j}$

$\vec{a}(t) = (6t-4)\hat{i} + 2\hat{j}$

$\boldsymbol{a}(t)$ (6 points) = $(6t-4)\hat{i}+2\hat{j}$

$\vec{v}(1) = (3-4+5)\hat{i} + (2+1)\hat{j} = 4\hat{i}+3\hat{j}$

$\|\vec{v}(1)\| = \sqrt{4^2+3^2} = \sqrt{25} = 5$

speed at $t=1$ (3 points) = 5

2. *(15 points)* Let $\boldsymbol{a}=\langle 2,-1,3\rangle$, $\boldsymbol{b}=\langle 1,3,2\rangle$ and $\boldsymbol{c}=\langle 0,1,-1\rangle$. Find each of the following.

(a) $2\boldsymbol{a}-3\boldsymbol{c}$ (3 points)

$2\vec{a} = \langle 4,-2,6\rangle$

$3\vec{c} = \langle 0,3,-3\rangle$

$2\vec{a} - 3\vec{c} = \langle 4,-2,6\rangle - \langle 0,3,-3\rangle$

$= \langle 4,-5,9\rangle$

$2\boldsymbol{a}-3\boldsymbol{c} =$ $\langle 4,-5,9\rangle$

(b) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$ (3 points)

$\vec{b}+\vec{c} = \langle 1,4,1\rangle$

$\vec{a}\cdot(\vec{b}+\vec{c}) = \langle 2,-1,3\rangle \cdot \langle 1,4,1\rangle$

$= 2-4+3 = 1$

$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) =$ 1

(c) projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ (6 points)

$\text{proj}_{\vec{b}}(\vec{a}) = \dfrac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b} = \dfrac{5}{14}\langle 1,3,2\rangle$

$\vec{a}\cdot\vec{b} = 2-3+6 = 5$

$\vec{b}\cdot\vec{b} = 1+9+4 = 14$

projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ = $\left\langle \frac{5}{14}, \frac{15}{14}, \frac{10}{14}\right\rangle$

(d) $\hat{\boldsymbol{a}}$ (the unit vector) (3 points)

$\hat{a} = \dfrac{\vec{a}}{\|\vec{a}\|} = \dfrac{1}{\sqrt{14}}\langle 2,-1,3\rangle$

$\|\vec{a}\| = \sqrt{2^2+(-1)^2+3^2}$

$= \sqrt{14}$

$\hat{\boldsymbol{a}} =$ $\dfrac{1}{\sqrt{14}}\langle 2,-1,3\rangle$

3. (20 points) For the points A(1, 3, 2), B(0, 3, 0) and C(2, 4, 3)

(a) *(10 points)* Find a normal vector to, and equation for, the plane through points A, B and C.

$\overrightarrow{BA} = \langle 1, 0, 2 \rangle$

$\overrightarrow{BC} = \langle 2, 1, 3 \rangle$

$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 2 & 1 & 3 \end{vmatrix} = -2\hat{i} + \hat{j} + \hat{k}$

$-2x + (y-3) + z = 0$

Normal vector = $\langle -2, 1, 1 \rangle$

Equation of plane: $-2x + y + z = 3$

(b) *(10 points)* Write a set of parametric equations for the line through point B and perpendicular to the plane in part (a).

$\vec{r}(t) = \langle -2, 1, 1 \rangle t + \langle 0, 3, 0 \rangle$

Line: $x(t) = -2t \quad y(t) = t + 3 \quad z(t) = t$

4. *(20 points)* Find the directional derivative of $f(x,y,z)=x^2y+y^2z+z^2x$ at $\boldsymbol{p}=(1,0,1)$ in the direction of $\boldsymbol{a}=\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$.

$$\frac{\partial f}{\partial x} = 2xy + z^2$$

$$\frac{\partial f}{\partial y} = x^2 + 2yz$$

$$\frac{\partial f}{\partial z} = y^2 + 2xz$$

$$\nabla f(1,0,1) = \langle 1, 1, 2 \rangle$$

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{\langle 1,2,2 \rangle}{\sqrt{1^2+2^2+2^2}} = \frac{\langle 1,2,2 \rangle}{3}$$

$$D_{\hat{a}} f(1,0,1) = \langle 1,1,2 \rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

$$= \frac{1}{3} + \frac{2}{3} + \frac{4}{3} = \frac{7}{3}$$

Answer: $\boxed{\frac{7}{3}}$

5. *(20 points)* Find all critical points of the function $f(x,y)=xy^2-6x^2-3y^2$. Determine if each critical point is a local minimum, a local max, or neither. If there are none of the given type of point, just write "None".

Polynomials are always differentiable, and there is no boundary, so all critical points are stationary points.

$\frac{\partial f}{\partial y} = 2xy - 6y \quad \frac{\partial f}{\partial x} = y^2 - 12x$ Want $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

$\Rightarrow x = \frac{y^2}{12} \quad \frac{y^3}{6} - 6y = 0 \Rightarrow y = 0, \pm 6$

$x = 0, 3$

So, critical points $(0,0), (3,6), (3,-6)$

$\frac{\partial^2 f}{\partial x^2} = -12 \quad \frac{\partial^2 f}{\partial y^2} = 2x - 6 \quad \frac{\partial^2 f}{\partial x \partial y} = 2y$

$D(x,y) = 72 - 24x - 4y^2 = f_{xx} f_{yy} - (f_{xy})^2$

$D(0,0) = 720, \ \frac{\partial^2 f}{\partial x^2} = -R < 0 \Rightarrow$ Max

$D(3,6) = -144 < 0 \Rightarrow$ Saddle

$D(3,-6) = -144 < 0 \Rightarrow$ Saddle

Critical Points: $(0,0), (3,6), (3,-6)$

Local Max: $(0,0)$

Local Min: None

~~Neither~~: $(3,6), (3,-6)$

Saddle

Indeterminate: None

6. (20 points) Calculate the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

Note – You must provide a correct calculation. Don't just state the answer.

$$\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dx\,dy \qquad \text{Convert to polar}$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta \qquad u = r^2 \quad du = 2r\,dr$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} e^{-u}\,du\,d\theta = \frac{1}{2}\int_0^{2\pi} -e^{-u}\Big|_0^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} d\theta = \frac{2\pi}{2} = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

Answer: $\sqrt{\pi}$

7. (20 points) Calculate the surface area of the part of the elliptic paraboloid $z=16-x^2-y^2$ above the xy-plane.

$\frac{\partial z}{\partial x} = -2x \quad \frac{\partial z}{\partial y} = -2y \qquad \iint_R \sqrt{1+(-2x)^2+(-2y)^2}\, dA$

$= \iint_R \sqrt{1+4x^2+4y^2}\, dA$

$\int_0^{2\pi}\int_0^4 \sqrt{1+4r^2}\, r\, dr\, d\theta$

$u = 1+4r^2$

$du = 8r\, dr$

$\Rightarrow \frac{1}{8}\int_0^{2\pi}\int_1^{65} \sqrt{u}\, du\, d\theta = \int_0^{2\pi} \frac{u^{3/2}}{12}\Big|_1^{65} d\theta$

$= \int_0^{2\pi} \left(\frac{65\sqrt{65}-1}{12}\right) d\theta = 2\pi\left(\frac{65\sqrt{65}-1}{12}\right)$

$= \pi\left(\frac{65\sqrt{65}-1}{6}\right)$

Surface Area: $\pi\left(\frac{65\sqrt{65}-1}{6}\right)$

8. (30 points) Calculate the following integrals:

a) $\int_0^{\frac{\pi}{2}} \int_0^{\sin y} e^x \cos y \, dx \, dy$ (10 points)

$$\int_0^{\pi/2} \int_0^{\sin y} e^x \cos y \, dx \, dy = \int_0^{\pi/2} e^x \cos y \Big|_0^{\sin y} dy$$

$$= \int_0^{\pi/2} (e^{\sin y} - 1) \cos y \, dy = \int_0^{\pi/2} e^{\sin y} \cos y \, dy - \int_0^{\pi/2} \cos y \, dy$$

$$= e^{\sin y} - \sin y \Big|_0^{\pi/2} = (e - 1) - (1 - 0) = e - 2$$

Answer: $\boxed{e-2}$

b) $\int_0^{\frac{\pi}{2}} \int_0^{z} \int_0^{y} \sin(x+y+z) \, dx \, dy \, dz$ (10 points)

$$= \int_0^{\pi/2} \int_0^{z} -\cos(x+y+z) \Big|_{x=0}^{x=y} dy \, dz$$

$$= \int_0^{\pi/2} \int_0^{z} [\cos(y+z) - \cos(2y+z)] \, dy \, dz$$

$$= \int_0^{\pi/2} \left[\sin(y+z) - \frac{\sin(2y+z)}{2} \Big|_0^{z} \right] dz$$

$$= \int_0^{\pi/2} \left[\left(\sin(2z) - \frac{\sin(3z)}{2} \right) - \left(\sin(z) - \frac{\sin(z)}{2} \right) \right] dz$$

$$= \int_0^{\pi/2} \left(\sin(2z) - \frac{\sin(3z)}{2} - \frac{\sin(z)}{2} \right) dz$$

$$= \frac{\cos(3z)}{6} + \frac{\cos(z)}{2} - \frac{\cos(2z)}{2} \Big|_0^{\pi/2}$$

$$= \left(0 + 0 - \left(-\tfrac{1}{2}\right)\right) - \left(\tfrac{1}{6} + \tfrac{1}{2} - \tfrac{1}{2}\right) = \tfrac{1}{2} - \tfrac{1}{6} = \tfrac{1}{3}$$

Answer: $\boxed{\frac{1}{3}}$

c) $\int_0^2 \int_x^2 6xe^{y^3} dy dx$ (10 points)

Switching the order of integration:

$$\int_0^2 \int_0^y 6xe^{y^3} dx dy$$

$$= \int_0^2 3y^2 e^{y^3} dy = e^{y^3}\Big|_0^2 = e^8 - 1$$

Answer: $e^8 - 1$

9. *(20 points)* Given $\boldsymbol{F}(x,y,z) = 5x^3yz\boldsymbol{i} - 2yx^2\boldsymbol{j} + y^3z^2\boldsymbol{k}$, calculate the following.

(a) div $\boldsymbol{F}$ (5 points)

$$15x^2yz - 2x^2 + 2y^3z$$

div $\boldsymbol{F}$ = $15x^2yz - 2x^2 + 2y^3z$

$F(x,y,z) = 5x^3yz\,\hat{i} - 2yx^2\,\hat{j} + y^3z^2\,\hat{k}$

(b) curl $\boldsymbol{F}$ (5 points)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5x^3yz & -2yx^2 & y^3z^2 \end{vmatrix} = (3y^2z^2)\,\hat{i} + (5x^3y - 0)\,\hat{j} + (-4xy - 5x^3z)$$

curl $\boldsymbol{F}$ = $\langle 3y^2z^2,\ 5x^3y,\ -4xy - 5x^3z \rangle$

(c) $\nabla(\nabla\cdot\boldsymbol{F})$ (5 points)

$\nabla(15x^2yz - 2x^2 + 2y^3z)$

$= \langle 30xyz - 4x,\ 15x^2z + 6y^2z,\ 15x^2y + 2y^3 \rangle$

$\nabla(\nabla\cdot\boldsymbol{F})$ = $\langle 30xyz - 4x,\ 15x^2z + 6y^2z,\ 15x^2y + 2y^3 \rangle$

(d) $\nabla\cdot(\nabla\times\boldsymbol{F})$ (5 points)

$= 0 + 5x^3 - 5x^3 = 0$

Note: In general, $\nabla\cdot(\nabla\times\vec{F}) = 0$.

$\nabla\cdot(\nabla\times\boldsymbol{F})$ = $\boxed{0}$

10. (30 points)

a) (20 points) Determine if the field $\boldsymbol{F}=(2xy+z^2)\boldsymbol{i}+x^2\boldsymbol{j}+(2xz+\pi\cos\pi z)\boldsymbol{k}$ is conservative. If it is conservative, find a function f such that $\nabla(f)=\boldsymbol{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^2 & x^2 & 2xz+\pi\cos\pi z \end{vmatrix}$$

$$= 0\,\hat{i} + (2z-2z)\hat{j} + (2x-2x)\hat{k} = \vec{0}$$

So, conservative.

$$\frac{\partial f}{\partial x} = 2xy+z^2 \Rightarrow f = x^2y + xz^2 + C(y,z)$$

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial C(y,z)}{\partial y} = x^2 \Rightarrow \frac{\partial C(y,z)}{\partial y} = 0 \Rightarrow$$

$$C(y,z) = C(z)$$

$$\frac{\partial f}{\partial z} = 2xz + C'(z) = 2xz + \pi\cos(\pi z)$$

$$\Rightarrow C(z) = \sin(\pi z) + C$$

$$f = x^2y + xz^2 + \sin(\pi z) + C$$

Conservative? (Circle One) (True) False

f (if it exists) = $x^2y + xz^2 + \sin(\pi z) + C$

Note: Any particular choice of C is also acceptable.

(This is problem 10 continued, so $\boldsymbol{F}=(2xy+z^2)\boldsymbol{i}+x^2\boldsymbol{j}+(2xz+\pi\cos\pi z)\boldsymbol{k}$).

b) (10 points) Calculate the line integral $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ where C is the line segment connecting the points (0,0,0) and (1,2,3).

$$\int_{(0,0,0)}^{(1,2,3)} F\cdot d\vec{r} = f(1,2,3) - f(0,0,0)$$

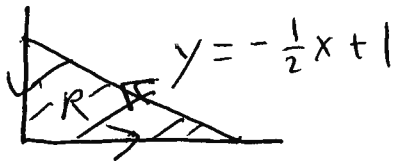
$$= (2+9+0) - (0+0+0)$$

$$= 11$$

Answer: $\boxed{11}$

11. (20 points) Calculate the line integral $\oint_C xy\,dx+(x+y)\,dy$ where C is the triangle with vertices (0,0), (2,0), and (0,1).

Hint – Use Green's theorem.



$$\frac{\partial N}{\partial x}=1 \qquad \frac{\partial M}{\partial y}=x$$

$$\oint_C xy\,dx+(x+y)\,dy \overset{\text{Green's Theorem}}{=} \iint_R (1-x)\,dA$$

$$= \int_0^2 \int_0^{-\frac{x}{2}+1} (1-x)\,dy\,dx = \int_0^2 (1-x)y\,\Big|_0^{1-\frac{x}{2}}\,dx$$

$$= \int_0^2 (1-x)\left(1-\frac{x}{2}\right)dx = \int_0^2 \left(1-\frac{3}{2}x+\frac{x^2}{2}\right)dx$$

$$= x-\frac{3}{4}x^2+\frac{x^3}{6}\,\Big|_0^2 = 2-3+\frac{8}{6}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Answer: $\boxed{\frac{1}{3}}$

12. (20 points) Calculate the surface integral $\int\int_G g(x,y,z)dS$ where $g=x+y$ and G is the part of $z=\sqrt{4-x^2}$ with $0 \le x \le \sqrt{3}$ and $0 \le y \le 1$.

Note - $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(\frac{x}{a})$.

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{4-x^2}} \quad \frac{\partial z}{\partial y} = 0$$

$$dS = \sqrt{1 + \frac{x^2}{4-x^2}}\, dA = \frac{2}{\sqrt{4-x^2}}$$

$$\Rightarrow 2\iint_R \frac{x+y}{\sqrt{4-x^2}}\, dA = 2\int_0^{\sqrt{3}}\int_0^1 \frac{x+y}{\sqrt{4-x^2}}\, dy\, dx$$

$$= 2\int_0^{\sqrt{3}} \frac{xy + \frac{y^2}{2}}{\sqrt{4-x^2}}\Big|_0^1 dx$$

$$= 2\int_0^{\sqrt{3}} \frac{x+\frac{1}{2}}{\sqrt{4-x^2}}\, dx = 2\int_0^{\sqrt{3}} \frac{x}{\sqrt{4-x^2}}\, dx + \int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$$

$$= -2\sqrt{4-x^2} + \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^{\sqrt{3}}$$

$$= \left(-2 + \frac{\pi}{3}\right) - (-4+0) = 2 + \frac{\pi}{3}$$

Answer: $2 + \frac{\pi}{3}$

Extra Credit:

(10 points) What did you think of the class, and what could be done to improve it?

Dylan is the best looking instructor ever!!!

Thank you all for a great class. Good luck in all you do.

-Dylan