Math2210 Midterm 3 Summer, 2010 Dylan Zwick

Name Solutions Date 7/29/2010

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. (15 points each) Evaluate the integrals.

(a) $\int_0^2 \int_1^3 x^2 y \, dy \, dx$

$$\int_0^2 \frac{x^2y^2}{2}\Big|_1^3 dx = \int_0^2 \left(\frac{9}{2}x^2 - \frac{x^2}{2}\right)dx$$

$$= \int_0^2 4x^2 dx = \frac{4x^3}{3}\Big|_0^2 = \frac{(4)(8)}{3} = \frac{32}{3}$$

Answer 1(a): $\boxed{\frac{32}{3}}$

(b) $\int_1^5 \int_0^x \frac{3}{x^2+y^2} dy \, dx$

$\cancel{= \int_1^5 3x}$

$$= \int_1^5 \frac{3}{x^2} \int_0^x \frac{1}{1+\left(\frac{y}{x}\right)^2} dy \, dx$$

$u = \frac{y}{x}$

$du = \frac{dy}{x}$

$x\,du = dy$

$$= \int_1^5 \frac{3}{x} \int_0^1 \frac{1}{1+u^2} du \, dx$$

$$= \int_1^5 \frac{3}{x} \tan^{-1}(u)\Big|_0^1 dx$$

$$= \cancel{\int_1^5 \frac{3}{x} \tan^{-1}(1) = 3\tan^{-1}(1)\ln(x)\Big|_1^5}$$

$$\cancel{= 3\tan^{-1}(1)\ln(5)}$$

$$= \int_1^5 \frac{3}{x}\tan^{-1}(1) = 3\tan^{-1}(1)\ln(x) = 3\tan^{-1}(1)\ln(5) = \frac{3\pi\ln(5)}{4}$$

Answer 1(b): $\boxed{\frac{3\pi\ln(5)}{4}}$

(Note: This is #1 continued!)

(c) $\int_0^{\pi/2}\int_0^{z}\int_0^{y} \sin(x+y+z)\,dx\,dy\,dz$

$$\int_0^{\pi/2}\int_0^{z} -\cos(x+y+z)\Big|_0^{y}\,dy\,dz$$

$$= \int_0^{\pi/2}\int_0^{z}\left[\cos(y+z) - \cos(2y+z)\right]dy\,dz$$

$$= \int_0^{\pi/2}\left[\left(\sin(y+z) - \frac{\sin(2y+z)}{2}\right)\right]\Big|_0^{z}\,dz$$

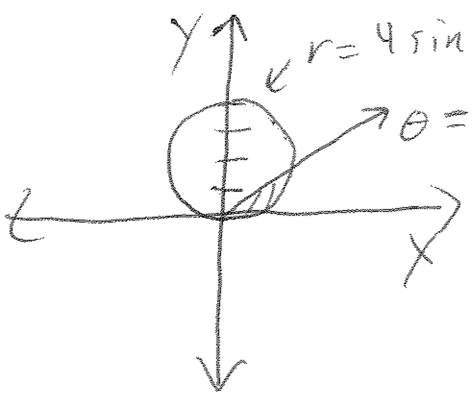
$$= \int_0^{\pi/2}\left[\left(\sin(2z) - \frac{\sin(3z)}{2}\right) - \left(\sin(z) - \frac{\sin(z)}{2}\right)\right]dz$$

$$= \frac{\cos(3z)}{6} + \frac{\cos(z)}{2} - \frac{\cos(2z)}{2}\Big|_0^{\pi/2} = \left(0+0-\left(-\tfrac{1}{2}\right)\right) - \left(\tfrac{1}{6}+\tfrac{1}{2}-\tfrac{1}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

Answer 1(c): $\boxed{1/3}$

2. (15 points) Find the area of the smaller region bounded by $\theta=\pi/6$ and $r=4\sin\theta$.



$$\cancel{\int_0^{\pi/6} 4r\sin}$$

$$\int_0^{\pi/6}\int_0^{4\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/6} 8\sin^2\theta\,d\theta$$

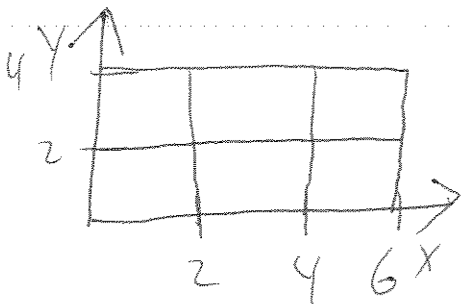
$$= \int_0^{\pi/6} 4(1-\cos 2\theta)\,d\theta$$

$$= 4\theta - 2\sin(2\theta)\Big|_0^{\pi/6}$$

$$= \left(\frac{2\pi}{3} - \sqrt{3}\right)$$

Answer 2: $\boxed{\frac{2\pi}{3} - \sqrt{3}}$

3. (15 points) If $R=((x,y): 0\leqslant x\leqslant 6, 0\leqslant y\leqslant 4)$ and P is the partition of R into six equal squares by the lines x = 2, x = 4, and y = 2. Approximate $\iint_R f(x,y)\,dA$ by calculating the corresponding Riemann sum $\sum_{k=1}^{6} f(\bar{x}_k, \bar{y}_k)\ \Delta\ A_k$, assuming that $(\bar{x}_k, \bar{y}_k)$ are the centers of the six squares. Take $f(x,y)=12-x-y$.



$f(1,1) = 10$ $\quad f(1,3) = 8$

$f(3,1) = 8$ $\quad f(3,3) = 6$

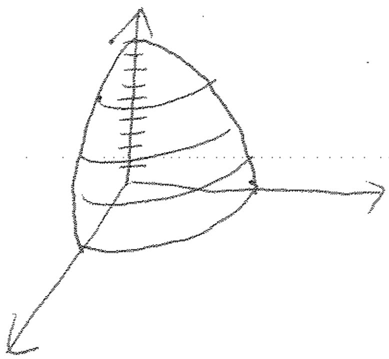
$f(5,1) = 6$ $\quad f(5,3) = 4$

$$\iint_R f(x,y)\,dA \approx (10+8+6+8+6+4)4$$

$$= \boxed{168}$$

Answer 3: ____________________

4. (20 points) Find the volume of the solid in the first octant bounded by the surface $z=9-x^2-y^2$ and the coordinate planes.



Convert to polar:

$$\int_0^{\pi/2}\int_0^3 (9-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{9r^2}{2} - \frac{r^4}{4} \Big|_0^3$$

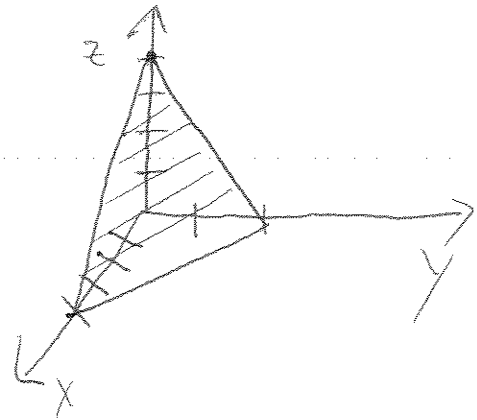
$$= \int_0^{\pi/2} \left(\frac{81}{2} - \frac{81}{4}\right) d\theta$$

$$= \int_0^{\pi/2} \frac{81}{4}\, d\theta = \frac{81\pi}{8}$$

Answer 4: $\frac{81\pi}{8}$

5. For the integral $V=\int_0^2\int_0^{4-2y}\int_0^{4-2y-z} dx\,dz\,dy$, do the following.

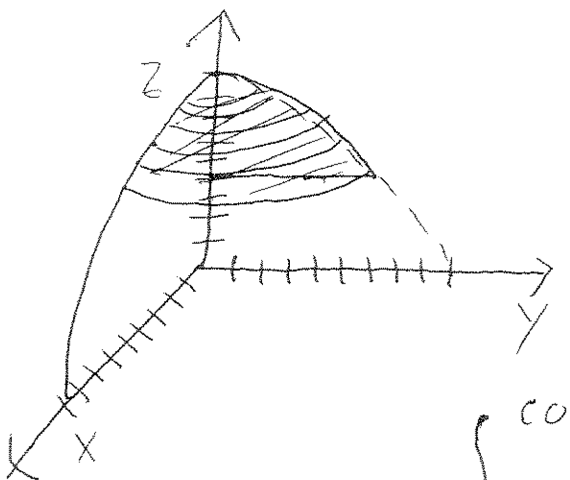
(a) (15 points) Rewrite it, changing the order of integration to $dz\,dy\,dx$. (Don't evaluate it, just set it up.)



$$V=\int_0^4\int_0^{2-\frac{x}{2}}\int_0^{4-2y-x} dz\,dy\,dx$$

Answer 5(a): $\int_0^4\int_0^{2-\frac{x}{2}}\int_0^{4-2y-x} dz\,dy\,dx$

(b) (15 points) For the integral $V = \int_0^{\sqrt{65}} \int_4^{\sqrt{81-y^2}} \int_0^{\sqrt{81-y^2-z^2}} dx\, dz\, dy$, rewrite it using spherical coordinates. (Don't evaluate it, just set it up.)



$$\int_0^{\cos^{-1}\left(\frac{4}{9}\right)} \int_0^{\pi/2} \int_{\frac{4}{\cos\phi}}^{9} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

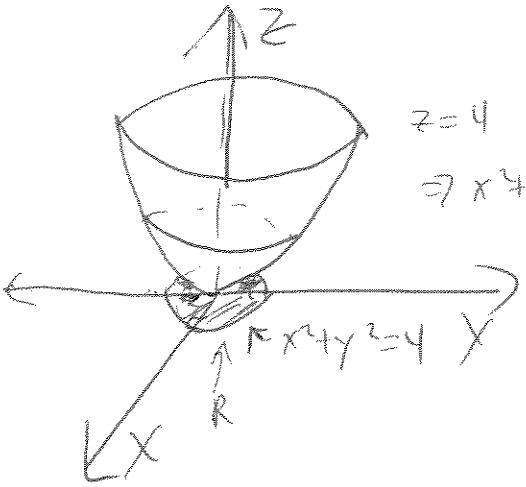
$z = 4$

$\rho \cos\phi = 4$

$\rho = \dfrac{4}{\cos\phi}$

Answer 5(b): $\int_0^{\cos^{-1}\left(\frac{4}{9}\right)} \int_0^{\pi/2} \int_{\frac{4}{\cos(\phi)}}^{9} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

6. (20 points) Find the surface area of the part of the paraboloid $z=x^2+y^2$ that is cut off by the plane $z=4$.



$z=4$
$\Rightarrow x^2+y^2=4$

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

$$SA = \iint_R \sqrt{1+(2x)^2+(2y)^2}\, dA$$

Polar

$$\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\, r\,dr\,d\theta$$

$u = 1+4r^2$
$du = 8r\,dr$

$$= \int_0^{2\pi}\int_1^{17} \frac{1}{8}\sqrt{u}\,du\,d\theta$$

$$= \int_0^{2\pi} \frac{u^{3/2}}{12}\Big|_1^{17} d\theta = \int_0^{2\pi}\left(\frac{17\sqrt{17}-1}{12}\right) d\theta$$

$$= \left(\frac{17\sqrt{17}-1}{6}\right)\pi$$

Answer 6: $\left(\frac{17\sqrt{17}-1}{6}\right)\pi$

7. (25 points) Find the maximum volume of a closed rectangular box with faces parallel to the coordinate planes inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad .$$

$V(x,y,z) = f(x,y,z) = 8xyz \qquad g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$\nabla f = \langle 8yz, 8xz, 8xy \rangle \quad \nabla g = \langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \rangle$

$\Rightarrow \quad 8yz = \frac{\lambda 2x}{a^2} \quad 8xz = \frac{\lambda 2y}{b^2} \quad 8xy = \frac{\lambda 2z}{c^2}$

$\lambda = \frac{4a^2yz}{x} \Rightarrow 8xz = \frac{8a^2y^2z}{xb^2} \quad z \neq 0$

$\Rightarrow \quad x^2 = \frac{a^2y^2}{b^2} \Rightarrow x = \frac{ay}{b}$

$\Rightarrow \frac{8ay^2}{b} = \frac{8a^2yz^2}{xc^2} \Rightarrow \frac{8ay^2}{b} = \frac{8a^2yz^2}{\left(\frac{ay}{b}\right)c^2}$

$\Rightarrow y^2 = \frac{b^2z^2}{c^2} \Rightarrow y = \frac{bz}{c} \Rightarrow x = \frac{az}{c}$

$\frac{a^2z^2}{a^2c^2} + \frac{b^2z^2}{b^2c^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{3z^2}{c^2} = 1$

$\Rightarrow z = \frac{c}{\sqrt{3}} \quad y = \frac{b}{\sqrt{3}} \quad x = \frac{a}{\sqrt{3}}$

$V = \frac{8abc}{3\sqrt{3}}$

Answer 7: $\boxed{\frac{8abc}{3\sqrt{3}}}$

8. Find the divergence and curl of the following vector fields. (15 points each)

a) $\boldsymbol{F}(x,y,z) = yz\,\boldsymbol{i} + xz\,\boldsymbol{j} + xy\,\boldsymbol{k}$

$$\nabla \cdot \vec{F} = \frac{\partial(yz)}{\partial y} + \frac{\partial(xz)}{\partial y} + \frac{\partial(xy)}{\partial z} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x-x)\,\hat{i} + (y-y)\,\hat{j} + (z-z)\,\hat{k} = \vec{0}$$

Divergence 8a): 0

Curl 8a): $0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$

b) $\boldsymbol{F}(x,y,z) = x^2\boldsymbol{i} - 2xy\,\boldsymbol{j} + yz^2\,\boldsymbol{k}$

$$\nabla \cdot \vec{F} = 2x - 2x + 2yz = 2yz$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -2xy & yz^2 \end{vmatrix} = z^2\,\hat{i} + 0\,\hat{j} - 2y\,\hat{k}$$

Divergence 8b): $2yz$

Curl 8b): $z^2\,\hat{i} - 2y\,\hat{k}$

Extra Credit (15 points): Calculate $\int_{-\infty}^{\infty} e^{-x^2}\,dx$. Note that you must show your calculation, you cannot just state the answer.

$$\left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}r\,dr\,d\theta$$

$u = +r^2$
$du = +2r\,dr$

$$= \int_0^{2\pi}\int_0^{\infty} \frac{e^{-u}}{2}\,du\,d\theta$$

$$= \int_0^{2\pi} -\frac{e^{-u}}{2}\Big|_0^{\infty} d\theta$$

$$= \int_0^{2\pi}\left(0-\left(-\tfrac{1}{2}\right)\right)d\theta$$

$$= \int_0^{2\pi}\frac{1}{2}\,d\theta = \frac{\theta}{2}\Big|_0^{2\pi} = \pi - 0 = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

Extra Credit Answer: $\boxed{\sqrt{\pi}}$