**Math2210 Midterm 2** Summer, 2010 Dylan Zwick

Name Solutions Date 7/21/2010

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. (25 pts) Find the directional derivative of $f(x,y,z)=x^3y-y^2z^2$ at $p=(-2,1,3)$ in the direction of $a=i-2j+2k$.

$$\nabla f = \langle 3x^2y,\ x^3-2yz^2,\ -2y^2z \rangle$$

$$\nabla f(-2,1,3) = \langle 12,\ -8-2(1)(3^2),\ -2(1^2)(3) \rangle$$

$$= \langle 12, -26, -6 \rangle$$

$$\|\vec{a}\| = \sqrt{1^2+(-2)^2+2^2} = \sqrt{9} = 3$$

$$\hat{a} = \left\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle$$

$$\hat{a}\cdot\nabla f(-2,1,3) = 12\left(\frac{1}{3}\right)+(-26)\left(-\frac{2}{3}\right)+(-6)\left(\frac{2}{3}\right)$$

$$= 4 + \frac{52}{3} - 4 = \frac{52}{3}$$

Answer: $52/3$

2. (25 pts) Show that the surfaces $x^2+4y+z^2=0$ and $x^2+y^2+z^2-6z+7=0$ are tangent to each other at (0,-1,2); that is, show that they have the same tangent plane at (0,-1,2).

Tangent plane of $x^2+4y+z^2$ is:

$\langle 2x, 4, 2z\rangle \quad (x=0,\ y=-1,\ z=2)$

$\langle 0, 4, 4\rangle \cdot \langle x, y+1, z-2\rangle = 0$

$4y+4+4z-8=0$

$4y+4z=4 \Rightarrow \cancel{x+} \boxed{y+z=1}$

Tangent plane of $x^2+y^2+z^2-6z+7=0$

$\langle 2x, 2y, 2z-6\rangle$ at $(0,-1,2)$

$\langle 0, -2, -2\rangle \cdot \langle x, y+1, z-2\rangle = 0$

$-2y-2-2z+4=0$

$-2y-2z=-2$

$\Rightarrow \boxed{y+z=1}$

Tangent plane ~~Answer:~~ Both $\cancel{x+}$ $y+z=1$.

3. (10 pts) Describe the largest set S on which $f(x, y, z)=\sqrt{(1-x^2-y^2-z^2)}$ is continuous.

Continuous where $1-x^2-y^2-z^2 \geq 0$

$\Rightarrow 1 \geq x^2+y^2+z^2$.

the set inside the unit sphere, including the sphere itself.

Answer : $x^2+y^2+z^2 \leq 1$

4. (25 pts) Find all critical <u>points</u> for $f(x,y)=xy^2-6x^2-3y^2$ . Determine whether each point is a minimum, maximum or saddle point.

$$\frac{\partial f}{\partial x} = y^2-12x \qquad \frac{\partial f}{\partial y} = 2xy-6y$$

$$y^2-12x=0 \qquad 2xy-6y=0$$

$$\Rightarrow x=\frac{y^2}{12} \qquad \Rightarrow \frac{y^3}{6}-6y=0$$

$$\Rightarrow y^3-36y=0 \Rightarrow (y^2-36)y=0$$

$$y=0,\pm 6 \qquad x=0,3$$

Critical points: $(0,0)$, $(3,6)$, $(3,-6)$

Note: No boundary, and always differentiable, so the ~~critic~~ stationary points are all the critical points.

$$\frac{\partial^2 f}{\partial x^2} = -2 \qquad \frac{\partial^2 f}{\partial y^2} = 2x-6 \qquad \frac{\partial^2 f}{\partial x \partial y} = 2y$$

$D(x,y) = 12-4x-4y^2$

$D(0,0)=12 \quad f_{xx}<0$ So, $(0,0)$ is a max.

$D(3,6)=D(3,-6)=-144$ So, a saddle.

Critical point(s) (Specify whether they're min, max or saddle.):

$(0,0)$ max, $(3,6)$ saddle, $(3,-6)$ saddle.

5. For $z = f(x, y) = -2x^2y^2 + \sin(\pi xy) + 5\ln(x+y)$, find

(a) (10 pts) $\dfrac{\partial z}{\partial y}$ at (2, -1)

$$\frac{\partial z}{\partial y} = -4x^2y + \pi x\cos(\pi xy) + \frac{5}{x+y}$$

$$\frac{\partial z}{\partial y}(2,-1) = -4(2^2)(-1) + 2\pi\cos(-2\pi) + \frac{5}{2+(-1)}$$
$$= 16 + 2\pi + 5$$
$$= 21 + 2\pi$$

Answer: $21 + 2\pi$

(b) (15 pts) $f_{xy}$

$$f_{xy} = -8xy - \pi^2 xy\sin(\pi xy) + \pi\cos(\pi xy) - \frac{5}{(x+y)^2}$$

Answer: $-8xy - \pi^2 xy\sin(\pi xy) + \pi\cos(\pi xy) - \dfrac{5}{(x+y)^2}$

6. For $f(x,y)=2e^{3y}\cos(2x)$ ,

(a) (10 pts) Find $\nabla f$ .

$$\nabla f = \langle f_x , f_y \rangle$$
$$= \langle -4e^{3y}\sin(2x), \ 6e^{3y}\cos(2x) \rangle$$

Answer : $\langle -4e^{3y}\sin(2x), 6e^{3y}\cos(2x) \rangle$

(b) (15 pts) Find the equation of the tangent plane at $(\pi/3, 0)$ .

$$f(\pi/3, 0) = 2e^{0}\cos\left(\tfrac{2\pi}{3}\right) = -1$$
$$f_x(\pi/3, 0) = -2\sqrt{3} \qquad f_y(\pi/3, 0) = -3$$

$$T(x,y) = -1 - 2\sqrt{3}(x - \pi/3) - 3y$$
$$= \frac{2\pi}{\sqrt{3}} - 1 - 2\sqrt{3}\,x - 3y$$

$$\Rightarrow \quad 2\sqrt{3}\,x + 3y + z = \frac{2\pi}{\sqrt{3}} - 1$$

Answer: $\boxed{2\sqrt{3}\,x + 3y + z = \frac{2\pi}{\sqrt{3}} - 1}$

7. (20 pts) Use the total differential $dz$ to approximate the change in z as (x, y) moves from P(-2, 4) to Q(-1.98,3.96) for $z = \ln(x^2 y)$

$$\frac{\partial z}{\partial x} = \frac{2xy}{x^2 y} = \frac{2}{x} \qquad \frac{\partial z}{\partial y} = \frac{x^2}{x^2 y} = \frac{1}{y}$$

$$\frac{\partial z}{\partial x}(-2,4) = -1 \qquad \frac{\partial z}{\partial y}(-2,4) = \frac{1}{4}$$

$$dx = -1.98 - (-2) = 0.02$$
$$dy = 3.96 - 4 = -0.04$$

$$dz = (-1)(0.02) + \left(\frac{1}{4}\right)(-0.04)$$
$$= -0.02 - 0.01 = -0.03$$

Answer: $\boxed{dz = -0.03}$

8. (20 pts) Find $\dfrac{\partial w}{\partial t}$ if $w(x,y)=\sqrt{x^2+y^2+z^2}$ , $x=\cos st$ , $y=\sin st$ , $z=s^2 t$ . (Your answer must be only in terms of $s$ and $t$.)

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial x}{\partial t} = -s \sin st \qquad \frac{\partial y}{\partial t} = s \cos st$$

$$\frac{\partial z}{\partial t} = s^2 \qquad x^2+y^2+z^2 = \cos^2 st + \sin^2 st + s^4t^2 = 1+s^4t^2$$

$$\Rightarrow \frac{\partial w}{\partial t} = \left(\frac{\cos st}{\sqrt{1+s^4t^2}}\right)\left(-s\sin(st)\right) + \left(\frac{\sin st}{\sqrt{1+s^4t^2}}\right)\left(s\cos(st)\right) + \frac{s^2t}{\sqrt{1+s^4t^2}}(s^2) = \frac{s^4t}{\sqrt{1+s^4t^2}}$$

Answer : $\dfrac{s^4 t}{\sqrt{1+s^4t^2}}$

9. Find the limit, if it exists. (Show all your reasoning.)

(a) $\lim_{(x,y)\to(1,2)} \dfrac{8x^2+3y^3}{x^2+y^2}$ (10 pts)

$1^2+2^2=5\neq 0$. Rational functions are continuous where the denominator is not 0.

So, $\lim_{(x,y)\to(1,2)} \dfrac{8(1^2)+3(2^3)}{1^2+2^2} = \dfrac{8+3(8)}{5} = \dfrac{32}{5}$

Answer : $\dfrac{32}{5}$

(b) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$ (15 pts)

Converting to polar

$\lim_{r\to 0} \dfrac{r^2\cos\theta\sin\theta}{r} = \lim_{r\to 0} r\cos\theta\sin\theta$

$-r \leq r\cos\theta\sin\theta \leq r$

$\lim_{r\to 0} -r = \lim_{r\to 0} r = 0$. So,

$\lim_{r\to 0} r\cos\theta\sin\theta = 0$.

Answer : $\boxed{0}$