Math2210 Midterm 1 Summer, 2010 Dylan Zwick

Name Solutions Date 7/16/2010

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. For $x=t^3-4t$ and $y=t^2-4$ such that $-3 \leq t \leq 3$, do the following:

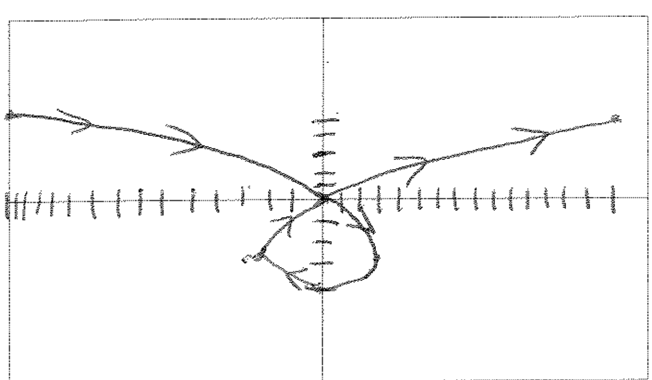
(a) (10 pts) Eliminate the parameter to obtain the corresponding Cartesian equation.

$$x = (t^2-4)t = yt \Rightarrow t = \frac{x}{y}$$

$$\Rightarrow y = \frac{x^2}{y^2} - 4 \Rightarrow y^3 + 4y^2 = x^2$$

Answer 1(a): $x^2 = (y+4)y^2$

(b) (10 pts) Graph the curve.



t	x	y
-3	-15	5
-2	0	0
-1	3	-3
0	0	-4
1	-3	-3
2	0	0
3	15	5

(c) (5 pts) Indicate if the curve is simple and/or closed.

Simple: T or F (circle one)

Closed: T or F (circle one)

2. (10 pts) Find the length of the curve given by $x=t+\frac{1}{t}$ and $y=\ln t^2$ for $1\le t\le 4$.

$$L=\int_1^4\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \qquad \frac{dx}{dt}=1-\frac{1}{t^2} \quad \frac{dy}{dt}=\frac{2}{t}$$

$$=\int_1^4\sqrt{\left(1-\frac{1}{t^2}\right)^2+\left(\frac{2}{t}\right)^2}\,dt$$

$$=\int_1^4\sqrt{\left(1+\frac{1}{t^2}\right)^2}\,dt=\int_1^4 1+\frac{1}{t^2}\,dt$$

$$=t-\frac{1}{t}\Big|_1^4=4-\frac{1}{4}=15/4$$

Answer 2: $15/4$

3. (15 pts) For position vector given by $r(t)=t^3\,\boldsymbol{i}+t^2+2t\,\boldsymbol{j}+\ln t\,\boldsymbol{k}$, find the velocity and acceleration vectors and the speed at $t=1$.

$v(t)$ = $3t^2\,\hat{i}+(2t+2)\,\hat{j}+\frac{1}{t}\,\hat{k}$

$a(t)$ = $6t\,\hat{i}+2\,\hat{j}-\frac{1}{t^2}\,\hat{k}$

$\vec{v}(1)=3\,\hat{i}+4\,\hat{j}+\hat{k}$

$\|\vec{v}(1)\|=\sqrt{3^2+4^2+1^2}=\sqrt{26}$

speed at $t=1$ = $\sqrt{26}$

4. (10 pts) Find the limit, if it exists. $\lim_{t\to 0}\left[\frac{3t^2\tan t}{2t^3}i - \frac{4t}{t^2-1}j + \frac{3t^2}{1-\cos^2 t}k\right]$

$$\lim_{t\to 0} \frac{3t^2 \tan t}{2t^3} = \frac{3}{2} \qquad \lim_{t\to 0} -\frac{4t}{t^2-1} = 0$$

$$\lim_{t\to 0} \frac{3t^2}{1-\cos^2 t} = \lim_{t\to 0} \frac{3t^2}{\sin^2 t} = 3$$

Answer (4) : $\frac{3}{2}\hat{i} + 3\hat{k}$

5. (10 pts) Find the equation of the sphere that has the line segment joining (0, 2, 3) and (4, 0, 5) as a diameter.

$$d = \sqrt{4^2 + (-2)^2 + (5-3)^2}$$
$$= \sqrt{16+4+4}$$
$$= \sqrt{24}$$
$$= 2\sqrt{6}$$

Radius = $\sqrt{6}$ units

$$m = \left(\frac{4}{2}, \frac{2}{2}, \frac{8}{2}\right)$$
$$= (2, 1, 4)$$

center = $(2, 1, 4)$

Eqn of sphere: $(x-2)^2 + (y-1)^2 + (z-4)^2 = 6$

6. (10 pts each) Let $a = \langle 2,0,3 \rangle$, $b = \langle -3,1,4 \rangle$ and $c = 5i + 2k$. Find each of the following.

(a) $2a - 3c$

$\langle 4, 0, 6 \rangle - \langle 15, 0, 6 \rangle$

$= \langle -11, 0, 0 \rangle$

$2a - 3c$ = $\langle -11, 0, 0 \rangle$

(b) $a \cdot (b + c)$

$\vec{b} + \vec{c} = \langle 2, 1, 6 \rangle$

$\vec{a} \cdot (\vec{b} + \vec{c}) = 2 \cdot 2 + 0 \cdot 1 + 3 \cdot 6$

$= 22$

$a \cdot (b + c)$ = 22

(c) $b \cdot c - |b|$

$\vec{b} \cdot \vec{c} = -15 + 8 = -7$

$|\vec{b}| = \sqrt{(-3)^2 + 1^2 + 4^2} = \sqrt{26}$

$= -7 - \sqrt{26}$

$b \cdot c - |b|$ = $-7 - \sqrt{26}$

(Note: This is # 6 continued.) $a=\langle 2,0,3\rangle$, $b=\langle -3,1,4\rangle$ and $c=5i+2k$

(d) $\hat{c}$ (the unit vector)

$$|\vec{c}| = \sqrt{25+4} = \sqrt{29}$$

$\hat{c}$ = $\frac{5}{\sqrt{29}}\hat{i} + \frac{2}{\sqrt{29}}\hat{k}$

(e) $a\times(b\times c)$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 4 \\ 5 & 0 & 2 \end{vmatrix} = 2\hat{i}+26\hat{j}-5\hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 3 \\ 2 & 26 & -5 \end{vmatrix} = -78\hat{i}+16\hat{j}+52\hat{k}$$

$a\times(b\times c)$ = $\langle -78, 16, 52\rangle$

(f) $a\cdot(b\times c)$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \langle 2,0,3\rangle\cdot\langle 2,26,-5\rangle$$
$$= 4-15 = -11$$

$a\cdot(b\times c)$ = -11

7. (10 pts each) For $a=2i-j+3k$ and $b=i+2j-k$, find each of the following:

(a) Direction cosines for a.

$\|\vec{a}\| = \sqrt{2^2+(-1)^2+3^2}$
$= \sqrt{14}$

$\cos\alpha =$ $2/\sqrt{14}$

$\cos\beta =$ $-1/\sqrt{14}$

$\cos\gamma =$ $3/\sqrt{14}$

(b) The angle θ between a and b. (Just write a simplified expression. If you don't have a calculator just write the numerical formula for the angle.)

$\|\vec{b}\| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$ $\quad \vec{a}\cdot\vec{b} = 2-2-3$
$= -3$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \frac{-3}{2\sqrt{21}}$$

$\theta =$ $\cos^{-1}\left(-\frac{3}{2\sqrt{21}}\right)$

(c) Find the projection of b onto a.

$$proj_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a} = \frac{-3}{14}\langle 2,-1,3\rangle$$

Projection of b onto a = $\left\langle -\frac{6}{14}, \frac{3}{14}, -\frac{9}{14}\right\rangle$

8. (10 pts each) For the planes given by $4x - y + 2z = 7$ and $5x + 3z = 13$, answer the following questions.

(a) Find the line of intersection between the planes and write that line in parametric equations.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 2 \\ 5 & 0 & 3 \end{vmatrix} = -3\hat{i} - 2\hat{j} + 5\hat{k}$$

Point on ~~plane~~ intersection $(5, 5, -4)$

Line: $x = 5 - 3t \quad y = 5 - 2t \quad z = -4 + 5t$

(b) Find the equation of the plane that is perpendicular to the line of intersection and goes through the point (0, 2, 1).

$$-3(x - 0) - 2(y - 2) + 5(z - 1) = 0$$

$$-3x - 2y + 5z + 4 - 5 = 0$$

$$-3x - 2y + 5z = 1$$

Equation of plane: $-3x - 2y + 5z = 1$

9. (a) (10 pts) Convert $2x^2+2y^2=5y+81$ from a Cartesian coordinate equation into an equation in cylindrical coordinates.

$$2(x^2+y^2) = 5y+81$$

$$2r^2 = 5r\sin\theta+81$$

$$r(2r-5\sin\theta)=81$$

Answer : $r(2r-5\sin\theta)=81$

(b) (10 pts) Convert $\rho=-3\sec\phi$ from a spherical coordinate equation into an equation in Cartesian coordinates.

$$\rho = -\frac{3}{\cos\phi} \Rightarrow \rho\cos\phi = -3$$

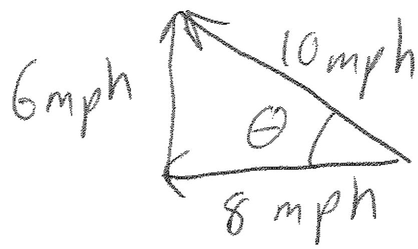
$$\Rightarrow z=-3$$

Answer : $z=-3$

Extra Credit: (10 pts)

A luxury cruiseliner is traveling due west at only 8 miles per hour. A woman on the ship is running across the ship, heading due north, at 6 miles per hour. What are the magnitude and direction of her velocity relative to the surface of the water? (If you don't have a calculator, just give the angle in simplified form.)

$\sqrt{6^2+8^2} = \sqrt{100} = 10$



$\theta = \tan^{-1}\left(\frac{6}{8}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

velocity magnitude: 10 mph

velocity direction: $\tan^{-1}\left(\frac{3}{4}\right)$ North of West