Math2210 Final Exam Summer, 2010 Dylan Zwick

Name Solutions Date 8/9/2010

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. *(15 points)* For position vector given by $\boldsymbol{r}(t)=(t^4-3t^2-t)\boldsymbol{i}+(t^3+t)\boldsymbol{j}$, find the velocity and acceleration vectors and the speed at $t=1$.

$\boldsymbol{v}(t) = (4t^3-6t-1)\hat{i}+(3t^2+1)\hat{j}$

$\boldsymbol{a}(t) = (12t^2-6)\hat{i}+(6t)\hat{j}$

$$\vec{v}(1) = (4(1)^3-6(1)-1)\hat{i}+(3(1^2)+1)\hat{j} = -3\hat{i}+4\hat{j}$$

$$\|\vec{v}(1)\| = \sqrt{(-3)^2+4^2} = 5$$

speed at $t=1$ = 5

2. *(20 points)* Let $a = \langle 1, -3, 2 \rangle$, $b = \langle 2, 6, 3 \rangle$ and $c = \langle -2, 5, 0 \rangle$. Find each of the following.

(a) $2a - 3c$

$$\langle 2, -6, 4 \rangle - \langle -6, 15, 0 \rangle = \langle 8, -21, 4 \rangle$$

$2a - 3c$ = $\langle 8, -21, 4 \rangle$

(b) $a \cdot (b + c)$

$$\langle 1, -3, 2 \rangle \cdot \langle 0, 11, 3 \rangle$$

$$= 0 - 33 + 6 = -27$$

$a \cdot (b + c)$ = -27

(c) projection of a onto b

$$\frac{\langle 1, -3, 2 \rangle \cdot \langle 2, 6, 3 \rangle}{(2^2 + 6^2 + 3^2)} \langle 2, 6, 3 \rangle$$

$$= \frac{2 - 18 + 6}{4 + 36 + 9} = -\frac{10}{49} \langle 2, 6, 3 \rangle = \left\langle -\frac{20}{49}, -\frac{60}{49}, -\frac{30}{49} \right\rangle$$

projection of a onto b = $\left\langle -\frac{20}{49}, -\frac{60}{49}, -\frac{30}{49} \right\rangle$

(d) $\hat{a}$ (the unit vector)

$$\| \vec{a} \| = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{14}$$

$\hat{a}$ = $\left\langle \frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right\rangle$

3. For the points A(-2, 4, 3), B(4, 2, 3) and C(1, 2, -1)

(a) *(10 points)* Write the equation of the plane through points A, B and C.

$$\vec{AB} = \langle 6, -2, 0 \rangle \qquad \vec{AC} = \langle 3, -2, -4 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 6 & -2 & 0 \\ 3 & -2 & -4 \end{vmatrix} = 8\hat{i} + 24\hat{j} - 6\hat{k}$$

$$8(x+2) + 24(y-4) \cancel{+} - 6(z-3) = 0$$

$$8x + 24y - 6z + 16 - 96 + 18 = 0$$

$$8x + 24y - 6z = 62 \Rightarrow 4x + 12y - 3z = 31$$

plane normal vector = $\langle 8, 24, -6 \rangle$

Equation of plane: $4x + 12y - 3z = 31$

(b) *(10 points)* Write a set of parametric equations for the line through point B and perpendicular to the plane in part (a).

$$x = 4 + 4t$$
$$y = 2 + 12t$$
$$z = 3 - 3t$$

Line: $x = 4 + 4t, \; y = 2 + 12t, \; z = 3 - 3t$

4. *(15 points)* Find the directional derivative of $f(x,y,z)=x^3y-y^2z^2+x^2y^2z^2$ at $p=(1,1,1)$ in the direction of $a=i+j+k$.

$$\nabla f = \langle 3x^2y + 2xy^2z^2,\ x^3 - 2yz^2 + 2x^2yz^2,\ -2y^2z + 2x^2y^2z \rangle$$

$$\nabla f(1,1,1) = \langle 5, 1, 0 \rangle$$

$$\hat{a} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$\nabla f(1,1,1) \cdot \hat{a} = \langle 5, 1, 0 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$= \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

Answer 4: $2\sqrt{3}$

5. *(20 points)* For the surface $F(x, y, z) = 4x^2 + 2xy + 2y^2 - 4yz - z^2 + x + z = 5$

(a) Find the equation of the tangent plane at the point (1, 1, 1).

$\nabla F = \langle 8x + 2y + 1,\ 2x + 4y - 4z,\ -4y - 2z + 1 \rangle$

$\nabla F(1,1,1) = \langle 11, 2, -5 \rangle$

$11(x-1) + 2(y-1) - 5(z-1) = 0$

$11x + 2y - 5z - 11 - 2 + 5 = 0$

$11x + 2y - 5z = 8$

Answer 5(a): $11x + 2y - 5z = 8$

(b) Find a point on the surface where the tangent plane is parallel to the plane $9x - 2y - z = 42$.

Note: The final exam had the plane $9x - y - z = 42$, which makes this problem very hard. Everybody received 10 points for this part.

For $9x - 2y - z = 42$ check:

$\nabla F(1,0,1) = \langle 9, -2, -1 \rangle$ and

$4(1^2) + 2(1)(0) + 2(0^2) - 4(0)(1) - 1^2 + 1 + 1$

$= 4 + 0 + 0 - 0 - 1 + 1 + 1 = 5$ ✓

Answer 5(b): $(1, 0, 1)$.

6. *(15 points)* Find all critical points of the function $f(x,y) = xy + \frac{2}{x} + \frac{2}{y}$.

Determine if each critical point is a local minimum, a local max, or neither. If there are none of the given type of point, just write "None".

$$\frac{\partial f}{\partial x} = y - \frac{2}{x^2} \qquad \frac{\partial f}{\partial y} = x - \frac{2}{y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{4}{x^3} \qquad \frac{\partial^2 f}{\partial y \partial x} = 1 \qquad \frac{\partial^2 f}{\partial y^2} = \frac{4}{y^3}$$

$$D(x,y) = \frac{16}{x^3 y^3} - 1$$

$$y - \frac{2}{x^2} = 0 \Rightarrow y = \frac{2}{x^2} \qquad x - \frac{2}{y^2} = 0$$

$$x \neq 0$$

$$x = \frac{2}{\left(\frac{2}{x^2}\right)^2} \Rightarrow \cancel{x^5} = x = \frac{x^4}{2} \Rightarrow 2 = x^3 \Rightarrow x = \sqrt[3]{2}$$

$$y = \frac{2}{2^{2/3}} = \sqrt[3]{2}$$

So, $(\sqrt[3]{2}, \sqrt[3]{2})$ is only critical point.

$$D(\sqrt[3]{2}, \sqrt[3]{2}) = \frac{16}{4} - 1 = 3.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{4}{2} = 2$$ So, local min.

Critical Points: $(\sqrt[3]{2}, \sqrt[3]{2})$

Local Max: None

Local Min: $(\sqrt[3]{2}, \sqrt[3]{2})$

Neither: None

7. (20 points) Find the minimum distance between the origin and the plane
$x+3y-2z=4$.

Lagrange Multipliers $f(x,y,z) = d^2(x,y,z) = x^2+y^2+z^2$

$g(x,y,z) = x+3y-2z-4$

$\nabla f = \langle 2x, 2y, 2z \rangle \quad \nabla g = \langle 1, 3, -2 \rangle$

$\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda \quad 2y = 3\lambda \quad 2z = -2\lambda$

$x = \frac{\lambda}{2}, \; y = \frac{3}{2}\lambda \quad z = -\lambda$

$\frac{\lambda}{2} + \frac{3}{\cancel{2}}\left(\frac{3}{2}\lambda\right) - 2(-\lambda) = 4$

$\Rightarrow \frac{\lambda}{2} + \frac{9}{2}\lambda + 2\lambda = 4 \quad \Rightarrow 7\lambda = 4 \quad \lambda = \frac{4}{7}$

$x = \frac{2}{7}, \; y = \frac{6}{7} \quad z = -\frac{4}{7}$

$d = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(-\frac{4}{7}\right)^2} = \sqrt{\frac{56}{49}} = \frac{2\sqrt{14}}{7}$

Other way:

$d = \frac{|Ax_0 + Bx_0 + Cx_0 - D|}{\sqrt{A^2+B^2+C^2}} = \frac{4}{\sqrt{1^2+3^2+(-2)^2}} = \frac{4}{\sqrt{14}} = \frac{2\sqrt{14}}{\cancel{2}\,7}$

Note: Either way would be fine here.

Answer: $\boxed{\frac{2\sqrt{14}}{7}}$

8. *(20 points)* Given $\boldsymbol{F}(x,y,z)=5x^3yz\,\boldsymbol{i}-2yx^2\,\boldsymbol{j}+y^3z^2\,\boldsymbol{k}$, calculate the following.

(a) div $\boldsymbol{F}$

$$15x^2yz-2x^2+2y^3z$$

div $\boldsymbol{F}$ = $15x^2yz-2x^2+2y^3z$

(b) curl $\boldsymbol{F}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5x^3yz & -2yx^2 & y^3z^2 \end{vmatrix} = (3y^2z^2)\hat{i}+(5x^3y)\hat{j}+(-4yx-5x^3z)\hat{k}$$

curl $\boldsymbol{F}$ = $3y^2z^2\,\hat{i}+5x^3y\,\hat{j}-(4xy+5x^3z)\hat{k}$

(c) $\nabla(\nabla\cdot\boldsymbol{F})$

$$(30xyz-4x)\hat{i}+(15x^2z+6y^2z)\hat{j}+(15x^2y+2y^3)\hat{k}$$

$\nabla(\nabla\cdot\boldsymbol{F})$ = $(30xyz-4x)\hat{i}+(15x^2z+6y^2z)\hat{j}+(15x^2y+2y^3)\hat{k}$

(d) $\nabla\cdot(\nabla\times\boldsymbol{F})$

$0+5x^3-5x^3=0.$

Note: Almost always 0.

$\nabla\cdot(\nabla\times\boldsymbol{F})$ = 0

9. *(15 points)* Evaluate the line integral $\int_C (x^2+y^2)\,ds$ given C is the path given by $x = e^t \sin t$, $y = e^t \cos t$ and $0 \le t \le 3$.

$$\frac{dx}{dt} = e^t \cos t + e^t \sin t \qquad \frac{dy}{dt} = e^t \cos t - e^t \sin t$$

$$\left(\frac{dx}{dt}\right)^2 = e^{2t}(\cos^2 t + 2\cos t \sin t + \sin^2 t)$$

$$\left(\frac{dy}{dt}\right)^2 = e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t)$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2e^{2t}(\cos^2 t + \sin^2 t) = 2e^{2t}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = e^t\sqrt{2}\,dt$$

$$\int_C (x^2+y^2)\,ds = \int_0^3 (e^{2t}\sin^2 t + e^{2t}\cos^2 t)\sqrt{2}\,e^t\,dt$$

$$= \int_0^3 \sqrt{2}\,e^{3t}\,dt = \frac{\sqrt{2}}{3} e^{3t}\Big|_0^3$$

$$= \frac{\sqrt{2}}{3}(e^9 - 1)$$

Answer 9: $\frac{\sqrt{2}}{3}(e^9 - 1)$

10. *(20 points)* Determine whether

$$F(x,y,z)=(3x^2y^2z^2+z^2+y)\mathbf{i}+(2x^3yz^2+2yz+x+1)\mathbf{j}+(2x^3y^2z+y^2+2xz)\mathbf{k}$$

is conservative. If so, find f such that $F=\nabla f$. If not, state that F is not conservative.

$$\frac{\partial f}{\partial x} = 3x^2y^2z^2+z^2+y \Rightarrow f = x^3y^2z^2+xz^2+xy+g(y,z)$$

$$\frac{\partial f}{\partial y} = 2x^3yz^2+x+\frac{\partial g}{\partial y} = 2x^3yz^2+2yz+x+1$$

$$\Rightarrow \frac{\partial g}{\partial y} = 2yz+1 \Rightarrow g(y,z) = y^2z+y+h(z)$$

$$\frac{\partial f}{\partial z} = 2x^3y^2z+2xz+y^2+h'(z) = 2x^3y^2z+y^2+2xz$$

$$h'(z)=0 \Rightarrow h(z)=C$$

$$f(x,y,z) = \cancel{x^3+y^2}\ \boxed{x^3y^2z^2+xz^2+xy+y^2z+y+C}$$

$$\nabla\times\vec{F} = \left((4x^3yz+2y)-(4x^3yz+2y)\right)\hat{i}$$
$$+\left((6x^2y^2z+2z)-(6x^2y^2z+2z)\right)\hat{j}$$
$$+\left((6x^2yz^2+1)-(6x^2yz^2+1)\right)\hat{k} = 0\hat{i}+0\hat{j}+0\hat{k}\ \checkmark$$

Conservative: True or False (circle one)

If conservative, f = $x^3y^2z^2+xz^2+xy+y^2z+y+C$

11. *(20 points)* Evaluate $\oint_C (x^2+4xy)\,dx+(2x^2+3y)\,dy$ where C is the ellipse $9x^2+16y^2=144$ oriented counter-clockwise.

$M = x^2+4xy \qquad N = 2x^2+3y$

$\dfrac{\partial M}{\partial y} = 4x \qquad \dfrac{\partial N}{\partial x} = 4x$

Green's theorem:

$$\iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \iint_S 0\, dA = \boxed{0}$$

Answer 11: $\boxed{0}$

Extra Credit:

(5 points) What is 4+9?

Answer : 13