Math2210 Final Exam Summer, 2009 Kelly MacArthur

Name Key Date ____________

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. *(10 points each)* Find each limit or state that it does not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2+1} \rightarrow \frac{0}{1} = 0$

limit: 0

(b) $\lim_{(x,y)\to(1,1)} \frac{x-2y+1}{x^2-y^2} \rightarrow \left(\frac{1-2+1}{1-1} = \frac{0}{0} \text{ case} \right)$

along $y=x^2$ $\lim_{x\to 1} \frac{x-2x^2+1}{x^2-x^4} \overset{L}{=} \lim_{x\to 1} \frac{1-4x}{2x-4x^3} = \frac{-3}{2-4} = \frac{3}{2}$

along $x=1$, $\lim_{y\to 1} \frac{2-2y}{1-y^2} \overset{L}{=} \lim_{y\to 1} \frac{-2}{-2y} = 1$ $\frac{3}{2} \neq 1$

limit: DNE

2. *(10 points each)* Let $a=\langle -1,0,5\rangle$, $b=\langle 2,2,3\rangle$ and $c=\langle -2,1,0\rangle$. Find each of the following.

(a) $2a-3c$

$\langle -2,0,10\rangle - \langle 6,6,9\rangle = \langle -8,-6,1\rangle$

$2a-3c$ = $\langle -8,-6,1\rangle$

(b) $a\cdot(b+c)$

$\langle -1,0,5\rangle \cdot \langle 0,3,3\rangle = 0+0+15$

$a\cdot(b+c)$ = 15

(c) projection of a onto b

$\hat{b} = \dfrac{\langle 2,2,3\rangle}{\sqrt{4+4+9}} = \langle \frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\rangle$

$\text{proj}_{\vec{b}}\vec{a} = (\vec{a}\cdot\hat{b})\hat{b}$

$= \left(\langle -1,0,5\rangle \cdot \langle \frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\rangle\right)\langle \frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\rangle$

$= \dfrac{-2+15}{\sqrt{17}}\langle \frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\rangle = \langle \frac{26}{17}, \frac{26}{17}, \frac{39}{17}\rangle$

projection of a onto b = $\langle \frac{26}{17}, \frac{26}{17}, \frac{39}{17}\rangle$

(d) $\hat{c}$ (the unit vector)

$\hat{c} = \dfrac{\langle -2,1,0\rangle}{\sqrt{4+1}} = \langle -2/\sqrt{5}, 1/\sqrt{5}, 0\rangle$

$\hat{c}$ = $\langle -2/\sqrt{5}, 1/\sqrt{5}, 0\rangle$

3. For the points A(-2, 3, 1), B(5, 1, 4) and C(1, 2, -1)

(a) *(10 points)* Write the equation of the plane through points A, B and C.

$\vec{AB} = \langle 7, -2, 3 \rangle$ $\vec{BC} = \langle -4, 1, -5 \rangle$

$$\vec{n} = \vec{AB} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -2 & 3 \\ -4 & 1 & -5 \end{vmatrix} = \hat{i}(10-3) - \hat{j}(-35+12) + \hat{k}(7-8)$$

$$= \langle 7, 23, -1 \rangle$$

$C(1,2,-1)$

$7x + 23y - z = D$

$7 + 46 + 1 = D$

$54 = D$

plane normal vector = $\langle 7, 23, -1 \rangle$

Eqn of plane: $7x + 23y - z = 54$

(b) *(10 points)* Write the parametric equations for the line through point C and perpendicular to the plane in part (a).

$C(1,2,-1)$ $\vec{n} = \langle 7, 23, -1 \rangle$

$x = 1 + 7t$

$y = 2 + 23t$

Line: $z = -1 - t$

4. *(10 points)* Find the directional derivative of $f(x,y,z)=y^3x+yz^2-2xz$ at $p=(-4,2,1)$ in the direction of $\boldsymbol{a}=2\boldsymbol{j}-\boldsymbol{k}$.

$\vec{a} = \langle 0, 2, -1 \rangle$

$\hat{a} = \langle \underset{a_1}{0}, \underset{a_2}{2/\sqrt{5}}, \underset{a_3}{-1/\sqrt{5}} \rangle$

$D_{\hat{a}} f = \hat{a} \cdot \nabla f = a_1 f_x + a_2 f_y + a_3 f_z$

$f_x = y^3 - 2z$

$f_x(-4,2,1) = 2^3 - 2$
$= 6$

$f_y = 3y^2x + z^2$

$f_y(-4,2,1) =$
$3(4)(-4) + 1$
$= -48 + 1$
$= -47$

$f_z = -2x + 2yz$

$f_z(-4,2,1)$
$= -2(-4) + 2(2)$
$= 8 + 4 = 12$

$D_{\hat{a}} f = 0(6) + \frac{2}{\sqrt{5}}(-47) + 12\left(\frac{-1}{\sqrt{5}}\right) = \frac{-94-12}{\sqrt{5}} = \frac{-106}{\sqrt{5}}$

Answer 4: $\frac{-106}{\sqrt{5}}$

5. *(20 points)* For the surface $F(x,y,z)=5x^2-2xy+3y^2+yz-z^2=6$

(a) Find the equation of the tangent plane at the point $(\sqrt{2},0,2)$.

$\nabla F = \langle 10x-2y,\ -2x+6y+z,\ y-2z \rangle$

$\nabla F(\sqrt{2},0,2) = \langle 10\sqrt{2}-0,\ -2\sqrt{2}+0+2,\ 0-4 \rangle$

$= \langle 10\sqrt{2},\ 2-2\sqrt{2},\ -4 \rangle = 2\langle 5\sqrt{2},\ 1-\sqrt{2},\ -2 \rangle$

$\vec{n} = \langle 5\sqrt{2},\ 1-\sqrt{2},\ -2 \rangle$ pt $(\sqrt{2},0,2)$

$5\sqrt{2}\,x+(1-\sqrt{2})y-2z=D$

$5\sqrt{2}(\sqrt{2})+(1-\sqrt{2})(0)-4=D$

$10-4=D$

$D=6$

Answer 5(a): $5\sqrt{2}\,x+(1-\sqrt{2})y-2z=6$

(b) Find a point on the surface where the tangent plane is parallel to the plane $8x+4y+z=7$.

$\nabla F = \langle 10x-2y,\ -2x+6y+z,\ y-2z \rangle = \langle 8,4,1 \rangle c$

① $10x-2y=8c$

② $-2x+6y+z=4c$

③ $y-2z=c$

$\boxed{y=c+2z}$

$\Rightarrow$ ① $10x-2(c+2z)=8c$

$10x-2c-4z=8c$

$10x=10c+4z$

$\boxed{x=c+\frac{2}{5}z}$

② $-2(c+\frac{2}{5}z)+6(c+2z)+z=4c$

$-2c-\frac{4}{5}z+6c+12z+z=4c$

$z(13-\frac{4}{5})=0$

$\Rightarrow z=0$

$\Rightarrow x=c$ and $y=c$

plug into surface

$\Rightarrow 5c^2-2c^2+3c^2+0-0=6$

$6c^2=6$

$c=\pm 1$

$\Rightarrow (1,1,0)$ or $(-1,-1,0)$

Answer 5(b): $(1,1,0)$ or $(-1,-1,0)$

6. *(15 points)* Find the minimum distance from the point (1, 2, 0) to the surface
$$z^2 = x^2 + y^2 .$$

minimize

$d = \sqrt{(x-1)^2 + (y-2)^2 + z^2} \Rightarrow f(x,y,z) = (x-1)^2 + (y-2)^2 + z^2$

such that (x,y,z) on $z^2 = x^2 + y^2$

$\Rightarrow g(x,y,z) = \boxed{x^2 + y^2 - z^2 = 0}$ ④

$\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-1), 2(y-2), 2z \rangle = \lambda \langle 2x, 2y, -2z \rangle$

① $2x - 2 = 2\lambda x$

$x - 1 = \lambda x$

$x(1-\lambda) = 1$

$\boxed{x = \frac{1}{1-\lambda}}$

(only if $\lambda \neq 1$, but if $\lambda = 1$, then

① $2x - 2 = 2x$

$\Leftrightarrow -2 = 0$

which is not true $\Rightarrow \lambda \neq 1$)

② $2y - 4 = 2\lambda y$

$y - 2 = \lambda y$

$y(1-\lambda) = 2$

$\boxed{y = \frac{2}{1-\lambda}}$

③ $2z = -2\lambda z$

$z = -\lambda z$

$z(1+\lambda) = 0$

$\boxed{z = 0}$ or $\boxed{\lambda = -1}$

$z = 0$:

④ $x^2 + y^2 = 0$

$\Rightarrow x = y = 0$

but if $x = 0$, then

① $-2 = 0$

which is not true,

$\Rightarrow x \neq 0$

N.S.

$\lambda = -1$:

① $x = \frac{1}{2}$

② $y = 1$

④ $\frac{1}{4} + 1 - z^2 = 0$

$z^2 = \frac{5}{4}$

$z = \frac{\pm\sqrt{5}}{2}$

$\boxed{(1/2, 1, \pm\sqrt{5}/2)}$

$\Rightarrow$ min distance:

$d = \sqrt{\left(\frac{1}{2} - 1\right)^2 + (1-2)^2 + \left(\sqrt{5}/2\right)^2} = \sqrt{\frac{1}{4} + 1 + \frac{5}{4}} = \frac{\sqrt{10}}{2}$

Answer 6: $\sqrt{10}/2$

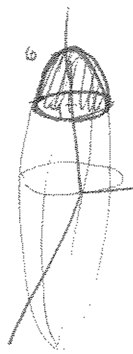
7. *(10 points each)* For the solid bounded by the surface $z=\sqrt{36-4x^2-4y^2}$ and the plane $z=4$, do the following.

(a) Set up the volume integral in Cartesian coordinates.

$z^2+4x^2+4y^2=36$

$\frac{x^2}{9}+\frac{y^2}{9}+\frac{z^2}{36}=1$

ellipsoid

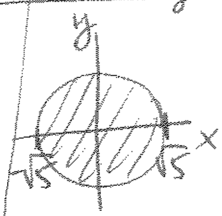


$4 \le z \le \sqrt{36-4x^2-4y^2}$

$z=4, \; 16+4x^2+4y^2=36$

$4x^2+4y^2=20$

$x^2+y^2 \le 5$



$-\sqrt{5-x^2} \le y \le \sqrt{5-x^2}$

$-\sqrt{5} \le x \le \sqrt{5}$

Answer 7(a): $\int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-x^2}}^{\sqrt{5-x^2}} \int_{4}^{\sqrt{36-4x^2-4y^2}} dz\,dy\,dx$

(b) Set up the integral using cylindrical coordinates.

$4 \le z \le \sqrt{36-4r^2}$

$0 \le r \le \sqrt{5}$

$0 \le \theta \le 2\pi$

Answer 7(b): $\int_0^{2\pi} \int_0^{\sqrt{5}} \int_4^{\sqrt{36-4r^2}} r\,dz\,dr\,d\theta$

(c) Evaluate the volume (using the integral of your choice).

$V = 2\pi \int_0^{\sqrt{5}} r \left(z \Big|_4^{\sqrt{36-4r^2}} \right) dr = 2\pi \int_0^{\sqrt{5}} r \left(\sqrt{36-4r^2}-4\right) dr$

$= 2\pi \int_0^{\sqrt{5}} r\sqrt{36-4r^2}\,dr - 2\pi\left(2r^2 \Big|_0^{\sqrt{5}}\right)$

$u = 36-4r^2$

$du = -8r\,dr$

$-\frac{1}{8}du = r\,dr$

$r=0, \; u=36$

$r=\sqrt{5}, \; u=16$

$= 2\pi \int_{36}^{16} \left(-\frac{1}{8}\right)\sqrt{u}\,du - 4\pi(5-0)$

$= -\frac{\pi}{4}\left(\frac{2}{3}\right)u^{3/2}\Big|_{36}^{16} - 20\pi = -\frac{\pi}{6}\left(16^{3/2}-36^{3/2}\right) - 20\pi$

$= -\frac{\pi}{6}(64-216) - 20\pi$

Answer 7(c): $16\pi/3$

$\begin{array}{r} 216 \\ -64 \\ \hline 152 \end{array}$

$\frac{\pi}{6}(152) - 20\pi = \pi\left(\frac{76}{3}-\frac{60}{3}\right) = \frac{16\pi}{3}$

8. *(10 points each)* Given $\boldsymbol{F}(x,y,z)=5x\,y^2z^2\boldsymbol{i}-y^3z^4\boldsymbol{j}+y\,x^3\boldsymbol{k}$, calculate the following.

(a) div $\boldsymbol{F}$ $= \nabla \cdot \vec{F} = \frac{\partial}{\partial x}\left(5xy^2z^2\right) + \frac{\partial}{\partial y}\left(-y^3z^4\right)$

$+ \frac{\partial}{\partial z}\left(yx^3\right)$

$= 5y^2z^2 - 3y^2z^4 + 0$

div $\boldsymbol{F}$ = $5y^2z^2 - 3y^2z^4$

(b) curl $\boldsymbol{F}$ $= \nabla \times \vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 5xy^2z^2 & -y^3z^4 & yx^3 \end{vmatrix} = \hat{i}\left(x^3 - (-4y^3z^3)\right) - \hat{j}\left(3yx^2 - 10xy^2z\right) + \hat{k}\left(0 - 10xyz^2\right)$$

curl $\boldsymbol{F}$ = $(x^3+4y^3z^3)\hat{i} + (10xy^2z - 3yx^2)\hat{j} + (-10xyz^2)\hat{k}$

(Note: This is #8 continued. $\boldsymbol{F}(x,y,z)=5x\,y^2z^2\boldsymbol{i}-y^3z^4\boldsymbol{j}+y\,x^3\boldsymbol{k}$)

(c) $\nabla(\nabla\cdot\boldsymbol{F}) = \nabla(\operatorname{div}\vec{F})$

$$\nabla\left(5y^2z^2-3y^2z^4\right) = 0\hat{i} + \left(10yz^2-6yz^4\right)\hat{j} + \left(10y^2z-12y^2z^3\right)\hat{k}$$

$\nabla(\nabla\cdot\boldsymbol{F}) = \underline{\left(10yz^2-6yz^4\right)\hat{j} + \left(10y^2z-12y^2z^3\right)\hat{k}}$

(d) $\nabla\cdot(\nabla\times\boldsymbol{F})$

$$= \nabla\cdot(\operatorname{curl}\vec{F}) = \operatorname{div}(\operatorname{curl}\vec{F})$$

$$= \frac{\partial}{\partial x}\left(x^3+4y^3z^3\right) + \frac{\partial}{\partial y}\left(10xy^2z-3yx^2\right) + \frac{\partial}{\partial z}\left(-10xyz^2\right)$$

$$= 3x^2 + 20xyz - 3x^2 - 20xyz$$

$\nabla\cdot(\nabla\times\boldsymbol{F}) = \underline{3x^2+20xyz-3x^2-20xyz}$

9. *(10 points)* Evaluate the line integral $\int_C \frac{1}{x^2+y^2} ds$, where C is the path given by $x=e^t \sin t$, $y=e^t \cos t$ and $0 \le t \le \ln 2$.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = e^t \sin t + e^t \cos t = e^t (\sin t + \cos t)$$

$$\frac{dy}{dt} = e^t \cos t - e^t \sin t = e^t (\cos t - \sin t)$$

$$\Rightarrow ds = \sqrt{e^{2t}(\sin t + \cos t)^2 + e^{2t}(\cos t - \sin t)^2}\, dt$$

$$= e^t \sqrt{(\sin^2 t + 2\sin t \cancel{\cos t} + \cos^2 t) + (\cos^2 t - 2\cos t \cancel{\sin t} + \sin^2 t)}\, dt$$

$$= \left(e^t \sqrt{1+1}\right) dt = \sqrt{2} e^t\, dt$$

$$\Rightarrow \int_C \frac{1}{x^2+y^2} ds = \int_0^{\ln 2} \frac{1}{e^{2t}(\sin^2 t + \cos^2 t)} (\sqrt{2} e^t)\, dt$$

$$= \int_0^{\ln 2} \sqrt{2} e^{-t}\, dt$$

$$= -\sqrt{2} e^{-t} \Big|_0^{\ln 2} = -\sqrt{2}\left(\frac{1}{e^t}\right)\Big|_0^{\ln 2}$$

$$= -\sqrt{2}\left(\frac{1}{2} - 1\right) = \frac{\sqrt{2}}{2}$$

Answer 9: $\sqrt{2}/2$

10. *(15 points)* Determine whether $\boldsymbol{F}(x,y,z)=(y^2e^x+6x+2yz)\boldsymbol{i}+(2ye^x+2xz+2yz^2)\boldsymbol{j}+(2xy+2y^2z-3z^2)\boldsymbol{k}$ is conservative. If so, find f such that $\boldsymbol{F}=\nabla f$. If not, state that $\boldsymbol{F}$ is not conservative.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2e^x+6x+2yz & 2ye^x+2xz+2yz^2 & 2xy+2y^2z-3z^2 \end{vmatrix}$$

$$= \hat{i}\left(2x+4yz-(2x+4yz)\right) - \hat{j}\left(2y-2y\right) + \hat{k}\left(2ye^x+2z-(2ye^x+2z)\right) = \vec{0}$$

Conservative: True or False (circle one)

$f_x = y^2e^x+6x+2yz \Rightarrow f = \int y^2e^x+6x+2yz\,dx$

① $f = y^2e^x+3x^2+2xyz+C(y,z)$

$\Rightarrow f_y = 2ye^x+2xz+C_y(y,z) = 2ye^x+2xz+2yz^2$

$\Rightarrow C_y(y,z) = 2yz^2$

$C(y,z) = \int 2yz^2\,dy$

$C(y,z) = y^2z^2 + D(z)$

$\Rightarrow$ ② $f = y^2e^x+3x^2+2xyz+y^2z^2+D(z)$

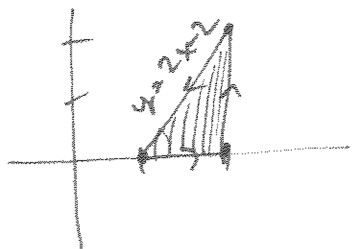
$\Rightarrow f_z = 2xy+2y^2z+D'(z) = 2xy+2y^2z-3z^2$

$D'(z) = -3z^2 \Rightarrow D(z) = \int -3z^2\,dz$

$= -z^3 + k$

If conservative, f = $y^2e^x+3x^2+2xyz+y^2z^2+-z^3+k$

11. *(10 points)* Use Green's Theorem to evaluate $\oint_C \overset{M}{(xy-1)}\,dx+\overset{N}{(\sin x+y^3)}\,dy$ where C is the boundary of the triangle with vertices (1, 0), (2, 0) and (2, 2) oriented counter-clockwise.



$m=\frac{2-0}{2-1}=2$

$y=2x+b$

$0=2+b$

$b=-2$

$y=2x-2$

$$\iint_S \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA$$

$$=\int_1^2\int_0^{2x-2}(\cos x - x)\,dy\,dx$$

$$=\int_1^2 (\cos x - x)\left(y\Big|_0^{2x-2}\right)dx$$

$$=\int_1^2 (\cos x - x)(2x-2)\,dx$$

$$=\int_1^2 2x\cos x - 2\cos x - 2x^2 + 2x\,dx$$

$$=(x\sin x+\cos x)\Big|_1^2 - 2\sin x\Big|_1^2 - \frac{2}{3}x^3\Big|_1^2 + x^2\Big|_1^2$$

$$=(\cancel{2\sin 2}+\cos 2)-(\sin 1+\cos 1)$$
$$-2(\cancel{\sin 2}-\sin 1)-\frac{2}{3}(8-1)+(4-1)$$

$$=\cos 2-\cos 1+\sin 1-\frac{14}{3}+3$$

$\int x\cos x\,dx$

$u=x \quad v=\sin x$

$du=dx \quad dv=\cos x\,dx$

$=x\sin x-\int \sin x\,dx$

$=x\sin x+\cos x+C$

Answer 11: $\cos 2-\cos 1+\sin 1-\frac{5}{3}$

12. *(10 points)* Evaluate the integral $\displaystyle\int_{\frac{\pi}{16}}^{\frac{\pi}{8}} \int_{csc(4x)}^{0} \int_{0}^{4xz} \sin\left(\frac{y}{z}\right) dy\, dz\, dx$.

$$\int_{\pi/16}^{\pi/8} \int_{\csc(4x)}^{0} -z \cos\left(\frac{y}{z}\right)\Big|_0^{4xz} dz\, dx$$

$$= \int_{\pi/16}^{\pi/8} \int_{\csc(4x)}^{0} -z\left(\cos(4x) - 0\right) dz\, dx$$

$$= \int_{\pi/16}^{\pi/8} -\cos(4x) \left(\int_{\csc(4x)}^{0} z\, dz\right) dx$$

$$= \int_{\pi/16}^{\pi/8} \frac{-\cos(4x)}{2} \left(z^2 \Big|_{\csc(4x)}^{0}\right) dx$$

$$= \frac{1}{2} \int_{\pi/16}^{\pi/8} \cos(4x) \left(\csc^2(4x)\right) dx$$

$$= \frac{1}{2} \int_{\pi/16}^{\pi/8} \frac{\cos(4x)}{\sin^2(4x)} dx$$

$u = \sin(4x)$

$du = 4\cos(4x)\, dx$

$\frac{1}{4} du = \cos(4x)\, dx$

$x = \pi/16, \quad u = \sqrt{2}/2$

$x = \pi/8, \quad u = 1$

$$= \frac{1}{2} \int_{\sqrt{2}/2}^{1} \frac{1}{4}\left(\frac{du}{u^2}\right)$$

$$= \frac{1}{8} \int_{\sqrt{2}/2}^{1} u^{-2}\, du = \frac{-1}{8}\left(\frac{1}{u}\Big|_{\sqrt{2}/2}^{1}\right)$$

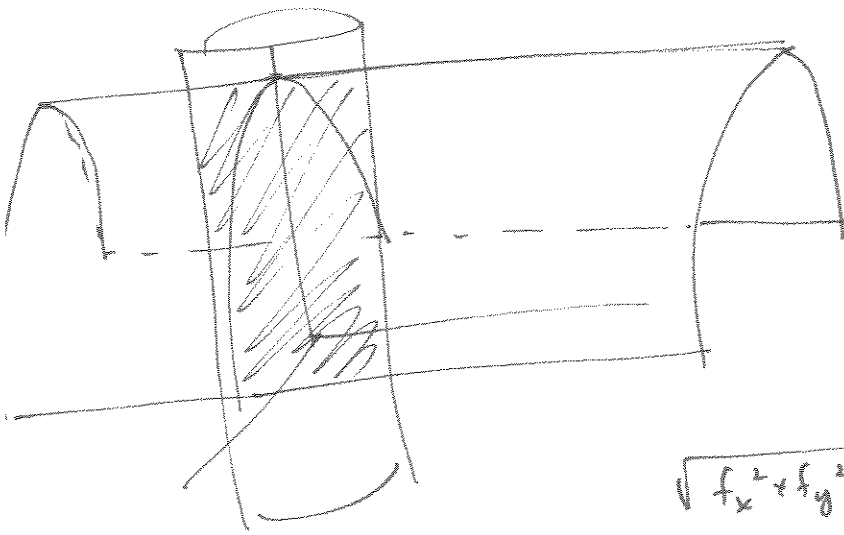
$$= \frac{-1}{8}\left(1 - \sqrt{2}\right)$$

Answer 12: $\dfrac{(\sqrt{2} - 1)}{8}$

13. *(10 points)* Set up the integral to find the area of the surface $x^2+z=5$ cut off by $x^2+y^2=9$ and xy-plane. (Don't bother evaluating the integral.)

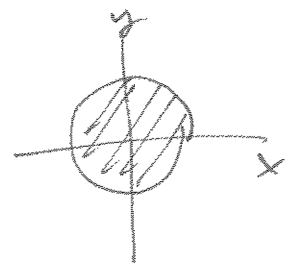
$z = 5 - x^2$

$f(x,y) = 5 - x^2$



$-\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$

$-3 \le x \le 3$



$f_x = -2x \qquad f_y = 0$

$\sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2+1}$

$$SA = \iint_S \sqrt{4x^2+1}\, dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4x^2+1}\, dy\, dx$$

$$\left(\text{or} \int_0^{2\pi} \int_0^3 \sqrt{4r^2\cos^2\theta + 1}\; r\, dr\, d\theta \right)$$

Answer: $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4x^2+1}\, dy\, dx$