Notes on Notation

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These notes address some common forms of mathematical notation that you're likely to see in this and future math classes, but you may not have seen before. I won't guarantee that these notes are exhaustive or free from typos, but I hope they help.

1 Set Theory

The basic object in mathematics is a set. A set is, loosely speaking, a collection of elements, and these elements can be "anything". I say anything in quotation marks because there are some caveats you have to worry about, but we don't need to discuss them here. For information about these caveats, look up Russel's paradox.

As an example of a set let's say that A is the set of positive even numbers less than 10. We'd write:

$$A = \{2, 4, 6, 8\}.$$

This indicates that A is the set consisting of the four elements $2, 4, 6, 8$. You can express a set by listing off its elements in parentheses. You can also use ellipsis if the number of elements in your set is large, and it's obvious what you mean. For example, if B is the set of positive even numbers less than 100000 we could write:

$$B = \{2, 4, 6, 8, \ldots, 99996, 99998\}.$$

Now, sets can also consist of an infinite number of elements, and we can represent this with a ellipsis at the end or the beginning (or both) of our enumeration of the elements. So, if C is the set of all positive even numbers we would write:

$$C = \{2, 4, 6, \ldots\}.$$

1.1 Set Relations

Now, with sets there are two main relations. One is a relation between a set and an element, and the other is a relation between a set and another set.

The relation between a set and an element is represented by $\in$, and can be loosely translated as the word "in". So, for example, $a \in A$ would mean that the element a is a member of the set A. In our set A constructed above, we could write $4 \in A$ to indicate 4 is one of the elements in A.

If we want to indicate an element is not in a set, we use the symbol $\notin$, which can be loosely translated as the words "not in". So if we write $b \notin A$, this means the element b is not in the set A. In our set A constructed above, we could write $5 \notin A$ to indicate 5 is not one of the elements in A.

The important relation between sets and other sets is the relation of inclusion. This relation is used to indicate if a set contains all the elements in another one. The symbol used for this relation is $\subset$. So, for example, if we write $A \subset B$ this would imply that every element in A is also in B. This would be true for our sets A and B constructed above. We would say that A is a "subset" of the set B. This notation can also go the other way, and we could write $B \supset A$, which just means the same thing, namely that A is a subset of B. It's kind of like being able to write both $<$ and $>$ when we're talking about inequality relations. We don't always have to make sure the smaller number, for example, is on the left. We could write $3 < 5$ or $5 > 3$. Both convey the same information.

Now, usually the symbol $\subset$ means that A is strictly contained in B. That means every element in A is in B, but not every element in B is in A. If we wanted to leave open the possibility that the two sets could be equal (could consist of exactly the same elements) we'd write $A \subseteq B$. This would means that A is a subset of B, and that A could possibly be equal to B. This relation would still be true of our sets A and B constructed above. Again, you can write it the other way around, $B \supseteq A$, and it means the same thing.

For example, if D represents the students over 20 years old in the algebra class I'll teach in the fall, and E represents the students in the class, I can write $D \subseteq E$. This indicates that D is a subset of E, but D could possibly be equal to E. In other words, it's possible all my students will be over 20 years old.

Now, it's not always the case that either $A \subseteq B$ or $B \subseteq A$. We could have F be the set of all even integers, and G be the set of all odd integers. In this case neither $F \subset G$ nor $G \subset F$ would be true.

1.2 Union, Intersection, and Difference

You can use sets to form other sets using the union, intersection, and difference operations. The union of two sets is the set consisting of all elements in either set. So, if we have the sets U and V then the union of U and V would be written as $U \cup V$. The $\cup$ symbol represents the union operation. So, for example, if:

$$U = \{1, 2, 3\}$$

and

$$V = \{2, 3, 4, 6\}$$

Then we'd have:

$$U \cup V = \{1, 2, 3, 4, 6\}.$$

$U \cup V$ is the set consisting of all elements in either U or V. Note that we would not list the elements 2 and 3 twice, as an element cannot appear in a set more than once.

The intersection of two sets is the set consisting of the elements in both. The symbol used for this operation is $\cap$. So, $U \cap V$ would be the intersection of U and V, and would represent the elements in both U and V. So,

$$U \cap V = \{2, 3\}.$$

The intersection of U and V would be the set consisting of just the elements 2 and 3.

Finally, the difference between two sets, say $U - V$, would be the set consisting of all the elements in U, except those that are also in V. So, for our example we'd have:

$$U - V = \{1\}.$$

The difference would be the set consisting of only the element 1. Now unlike union and intersection the difference operation is not symmetric, and $U - V$ is not necessarily the same as $V - U$. In our example we'd have:

$$V - U = \{4, 6\}.$$

This is clearly not the same as $U - V$.

1.3 Special Sets

There are some sets that come up so often we give them their own special symbols. One of these is the integers, to which we give the symbol $\mathbb{Z}$. The Z is for "zahlen", the German word for "numbers". The font is called "blackboard bold", and it's used for most of these number sets. So,

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

If we want to represent the non-negative integers we'd use the symbol $\mathbb{Z}^+$. So,

$$\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}.$$

If we want to represent just the positive integers we'd use the symbol $\mathbb{P}$. So,

$$\mathbb{P} = \{1, 2, 3, \ldots\}.$$

Note that this symbol is not used very often, and its meaning is not as universal as the other symbols mentioned here.

Finally, as you might imagine, the symbol for the nonpositive integers is $\mathbb{Z}^-$. I'm unaware of any symbol for the strictly negative integers, but you could write them as $\mathbb{Z}^- - \{0\}$.

Now, a rational number is a number that can be written as one integer divided by another. The set of rational numbers is represented as $\mathbb{Q}$. The use of the letter Q is because it's a "*q*uotient" of two integers. The set of nonnegative rational numbers is represented as $\mathbb{Q}^+$, and the set of nonpositive rational numbers is represented as $\mathbb{Q}^-$. So, using the notation we've learned so far we'd say:

$$r \in \mathbb{Q} \text{ means that } r = \frac{a}{b} \text{ with } a, b \in \mathbb{Z}.$$

The set of real numbers is represented by $\mathbb{R}$, while the set of nonnegative real numbers is represented by $\mathbb{R}^+$, and the set of nonpositive real numbers is represented by $\mathbb{R}^-$. I'll let you figure out why the letter R is used for *r*eal numbers. The set of complex numbers is represented by $\mathbb{C}$. For complex numbers the idea of positive and negative doesn't make as much sense, and so we wouldn't write $\mathbb{C}^+$, for example, as it's unclear what that would mean.

Finally, it's important to note that we can talk about the set consisting of no elements. This is called the *empty set*, and it's denoted by the symbol $\emptyset$.

In our earlier example we said that we'd call F the set of all even integers, and G the set of all odd integers. In this case we'd write:

$$F \cap G = \emptyset.$$

There are no integers that are both odd and even, and so the intersection of F and G would be empty.

2 Logic

There are a few symbols and ideas from formal logic that are used all the time in mathematics. One of them is the "for every" or "for all" symbol, which is an upside-down A: $\forall$. Another is the "there exists" symbol, which is a backwards E: $\exists$. If we want to indicate that the existance is unique, "there exists a unique", we'd put an exclamation mark after the "there exists" symbol: $\exists!$.

For example, the statement:

$$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} \text{ such that } y > x$$

says that for any real number x there exists another real number y that is greater than x.

As another example, the statement:

$$\forall x \in \mathbb{Z}\ \exists! y \in \mathbb{Z} \text{ such that } y = x + 1$$

says that for any integer x there exists a unique integer y such that y is one more than x.

If we have a right arrow, either with two lines or with one: $\rightarrow$ or $\Rightarrow$, this just means that the statement on the right follows logically from the statement on the left. Occasionally if the arrow begins a line it will mean the statement on the right follows logically from the statement above it. Similarly, if we have a left arrow: $\leftarrow$ or $\Leftarrow$, this just means that the statement on the left follows logically from the statement on the right.

So, for example, the statement:

$$x > 3 \Rightarrow x > 2$$

says that if x is greater than 3 it follows that x is greater than 2.

Finally, you will occasionally see at the conclusion of a proof the letters "Q.E.D." These letters stand for the Latin phrase "quad erat demonstrandum", which translates as "which was to be demonstrated." It means the proof is complete. In many modern math textbooks the end of a proof is marked with a square: $\square$. The symbol is sometimes called a "tombstone" or a "halmos". The "halmos" terminology comes from the mathematician Paul Halmos, who first started using it to mark the end or proofs.

So, for example, we could write:

Theorem - The number of prime numbers is infinite.

PROOF - Suppose the number of prime numbers is finite, and enumerate them as $p_1, p_2, \ldots, p_n$. Construct a number n by multiplying all the prime numbers together and adding one:

$$n = p_1 p_2 \ldots p_n + 1.$$

This number n cannot be a prime number, as it must be larger than all the prime numbers. So n must be composite, and it's therefore divisible by a prime. Call such a prime p_k. The first term in the definition of n would be divisible by p_k by construction, but the second term would be $1/p_k$, which is not an integer and must be less than 1. So, n/p_k could not be an integer, and therfore n is not divisible by p_k. This is a contradiction, and so our initial assumption, that there are a finite number of primes, must be false. $\square$