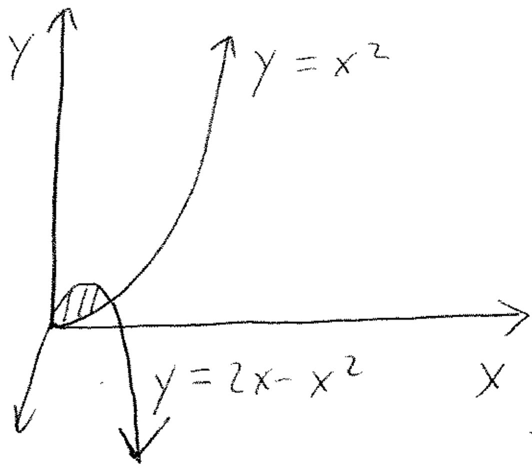
Math1210 Final Exam Summer, 2009 Dylan Zwick

Name Dylan Zwick (Solutions) Date 06/10/09

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

Part 1: Do all of the following 7 problems.

1. (15 points) Find the area of the region bounded by the curves $y=x^2$ and $y=2x-x^2$.



$x^2 = 2x - x^2$

$\Rightarrow 2x^2 = 2x \Rightarrow x^2 = x$

$\Rightarrow x = \{0, 1\}$

$$\int_0^1 [(2x-x^2) - x^2]\,dx$$

$$= \int_0^1 (2x-2x^2)\,dx = x^2 - \frac{2}{3}x^3\Big|_0^1$$

$$= \left(1-\frac{2}{3}\right) - (0-0) = \frac{1}{3}$$

Area: $\boxed{\frac{1}{3}}$

2. (15 points) Find the equation of the tangent line to the curve $f(x) = \frac{5\sqrt{x^3+1}-2x}{\cos x}$ at $x=0$.

$$f'(x) = \frac{\cos x\left(\frac{15x^2}{2\sqrt{x^3+1}} - 2\right) + \left(5\sqrt{x^3+1} - 2x\right)\sin x}{\cos^2 x}$$

$$\Rightarrow f'(0) = \frac{1(0-2) + (5)(0)}{1^2} = -2$$

$f(0) = 5$

Tangent Line: $\boxed{y = -2x + 5}$

3. (10 points each) Find each limit, if it exists.

(a) $\lim_{x \to 0} \dfrac{\sin(4x)}{\tan(5x)\cos x}$ = $\boxed{4/5}$

$$= \lim_{x \to 0} \frac{\sin(4x)}{\frac{\sin(5x)}{\cos(5x)} \cos x} = \lim_{x \to 0} \frac{4x}{\frac{5x}{1} \cdot 1} = \frac{4}{5}$$

(b) $\lim_{x \to 5} \dfrac{2x^2 - x - 45}{3x^2 - 9x - 30}$ = $\boxed{19/21}$

$$\lim_{x \to 5} \frac{2x^2 - x - 45}{3x^2 - 9x - 30} = \lim_{x \to 5} \frac{\cancel{(x-5)}(2x+9)}{\cancel{(x-5)}(3x+6)}$$

$$= \lim_{x \to 5} \frac{2x+9}{3x+6} = \frac{19}{21}$$

(c) $\lim_{x \to \infty} \dfrac{\sqrt{2x^5} + x^2 - 4x + 1}{-3x^2 + x + 5}$ = $\boxed{-\infty}$

$$= \lim_{x \to \infty} \frac{\sqrt{2}\, x^{5/2} + x^2 - 4x + 1}{-3x^2 + x + 5} \to -\infty$$

4. (10 points each) Evaluate these integrals.

(a) $\int \frac{y^5+2y^2-1}{\sqrt{y}}\,dy = \boxed{\frac{2}{11}y^{11/2}+\frac{4}{5}y^{5/2}-2y^{1/2}+C}$

$$= \int \left(y^{9/2}+2y^{3/2}-y^{-1/2}\right)dy$$

$$= \frac{2y^{11/2}}{11}+\frac{4y^{5/2}}{5}-2y^{1/2}+C$$

(b) $\int_{-5}^{5} x^3\cos(3x)\,dx = \boxed{0}$

$$(-a)^3\cos(3(-a)) = -a^3\cos(3a)$$

So, $x^3\cos(3x)$ is odd.

$$\Rightarrow \int_{-5}^{5} x^3\cos(3x)\,dx = 0$$

(c) $\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{(\cos\theta+1)^3}\,d\theta = \boxed{3/8}$

$u = \cos\theta + 1$

$du = -\sin\theta\, d\theta$

$$\Rightarrow \int_2^1 \frac{-du}{u^3} = \int_1^2 \frac{1}{u^3}\,du$$

$$= -\frac{1}{2u^2}\Big|_1^2 = -\frac{1}{2}\left(\frac{1}{4}-1\right) = -\frac{1}{2}\left(-\frac{3}{4}\right) = \frac{3}{8}$$

(Note: This is #4 continued!)

(d) $\int 5t^3\cos(t^4-1)\,dt = \frac{5}{4}\sin(t^4-1)+C$

$u = t^4-1$

$du = 4t^3\,dt \Rightarrow \int \frac{5}{4}\cos(u)\,du$

$\Rightarrow \frac{5}{4}du = 5t^3\,dt \quad = \frac{5}{4}\sin(u)+C$

$= \frac{5}{4}\sin(t^4-1)+C$

(e) $\int_1^{81} \frac{1}{\sqrt[4]{16x^7}}\,dx = \boxed{\frac{52}{81}}$

$= \int_1^{81} \frac{1}{2\,x^{7/4}}\,dx = \frac{1}{2}\int_1^{81} x^{-7/4}\,dx$

$= \frac{1}{2}\left(\frac{x^{-3/4}}{\left(-\frac{3}{4}\right)}\right) = -\frac{2}{3}\,x^{-3/4}\Big|_1^{81}$

$= -\frac{2}{3}\left(\frac{1}{27}-1\right) = -\frac{2}{3}\left(-\frac{26}{27}\right) = \boxed{\frac{52}{81}}$

5. (10 points each) Find the indicated derivative of the given functions.

(a) $\frac{d}{dx}\left(\frac{5\tan x - 4x^{\frac{2}{3}}}{x^4+2x^2-1}\right)$ (Do **not** bother to simplify!)

$$= \frac{(x^4+2x^2-1)(5\sec^2 x - \frac{8}{3}x^{-1/3}) - (5\tan x - 4x^{2/3})(4x^3+4x)}{(x^4+2x^2-1)^2}$$

$$\frac{d}{dx}\left(\frac{5\tan x - 4x^{\frac{2}{3}}}{x^4+2x^2-1}\right) = \frac{(x^4+2x^2-1)(5\sec^2 x - \frac{8}{3}x^{-1/3}) - (5\tan x - 4x^{2/3})(4x^3+4x)}{(x^4+2x^2-1)^2}$$

(b) $D_x((x^3+5x)^6 \sec x)$ (Do **not** bother to simplify!)

$$= (x^3+5x)^6 \sec x \tan x + 6(x^3+5x)^5(3x^2+5)\sec x$$

$$= (x^3+5x)^6 \sec x \tan x + 6(3x^2+5)(x^3+5x)^5 \sec x$$

$$D_x((x^3+5x)^6 \sec x) = (x^3+5x)^6 \sec x \tan x + 6(3x^2+5)(x^3+5x)^5 \sec x$$

(Note: This is #5 continued!)

(c) $D_x(\sqrt{4x^5+\sqrt[3]{x^4+3x}})$ (Do **not** bother to simplify!

$$=\left(\frac{1}{2\sqrt{4x^5+\sqrt[3]{x^4+3x}}}\right)\left(4(5x^4)+\frac{1}{3(x^4+3x)^{2/3}}(4x^3+3)\right)$$

$$=\left(\frac{1}{2\sqrt{4x^5+(x^4+3x)^{1/3}}}\right)\left(20x^4+\frac{(4x^3+3)}{3(x^4+3x)^{2/3}}\right)$$

$D_x(\sqrt{4x^5+\sqrt[3]{x^4+3x}}) =$ $\left(\frac{1}{2\sqrt{4x^5+(x^4+3x)^{1/3}}}\right)\left(20x^4+\frac{4x^3+3}{3(x^4+3x)^{2/3}}\right)$

(d) $\frac{dy}{dx}$ given $2\sqrt{xy}+y^4=5x^2+1$

$$2\sqrt{x}\left(\frac{1}{2\sqrt{y}}\right)dy+2\left(\frac{1}{2\sqrt{x}}\right)\sqrt{y}\,dx+4y^3dy=10x\,dx$$

$$\Rightarrow \left(\sqrt{\frac{x}{y}}+4y^3\right)dy=\left(10x-\sqrt{\frac{y}{x}}\right)dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{10x-\sqrt{\frac{y}{x}}}{\sqrt{\frac{x}{y}}+4y^3}$$

$\frac{dy}{dx} =$ $\frac{10x-\sqrt{\frac{y}{x}}}{\sqrt{\frac{x}{y}}+4y^3}$

6. (5 points each) For $f(x)=\frac{2(x+1)^3}{(x-2)^3}$

(given $f'(x)=\frac{-18(x+1)^2}{(x-2)^4}$ and $f''(x)=\frac{36(x+1)(x+4)}{(x-2)^5}$).

(a) Find the asymptotes. (x-value)

Vertical asymptotes: $X = 2$

(b) Fill in the sign line for $f'(x)$.

	$-$		$-$		$-$
$f'(x)$		-1		2	

(c) Find all local minimum and maximum **points**. (x and y values)

Local Max points: None

Local Min points: None

(d) Fill in the sign line for $f''(x)$.

	$-$		$+$		$-$		$+$
$f''(x)$		-4		-1		2	

(e) Find all inflection **points**. (x and y values)

$$\frac{2(-3)^3}{(-6)^3} = \frac{1}{4}$$

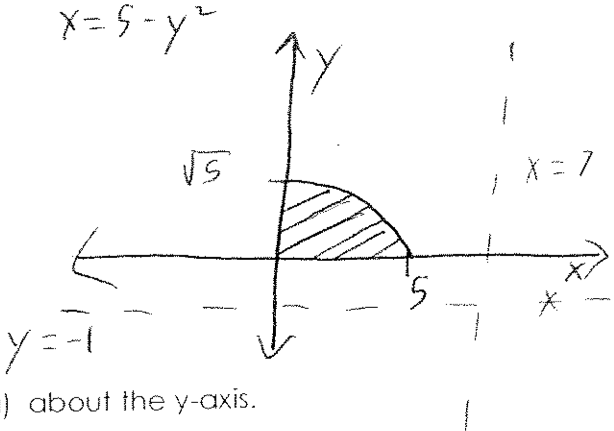
Inflection points: $\left(-4, \frac{1}{4}\right)$, $(-1, 0)$

7. (10 points each) **Setup** (you do NOT have to evaluate) the integrals for the volume of the solid generated by revolving the region bounded by

~~$x=5+y^2$~~ , $x=0$, and $y=0$

$X = 5 - y^2$

$y = \sqrt{5-x}$



(a) about the y-axis.

V = $\pi \int_0^{\sqrt{5}} (5-y^2)^2\,dy$ or $2\pi \int_0^5 x\sqrt{5-x}\,dx$

(b) about the x-axis.

V = $2\pi \int_0^{\sqrt{5}} y(5-y^2)\,dy$ or $\pi \int_0^5 (\sqrt{5-x})^2\,dx$

(c) about the line x = 7.

$2\pi \int_0^5 (7-x)\sqrt{5-x}\,dx$ or

V = $\pi \int_0^{\sqrt{5}} [\,7^2 - (7-(5-y^2))^2\,]\,dy$

$= \pi \int_0^{\sqrt{5}} (7^2 - (2+y^2)^2)\,dy$

(d) about the line y = -1.

$\pi \int_0^5 [(1+\sqrt{5-x})^2 - 1^2]\,dx$ or

V = $2\pi \int_0^{\sqrt{5}} (y+1)(5-y^2)\,dy$

Part 2: (20 points each) **Choose 2 out of the next 4 questions to do. Indicate clearly which problems you want graded!!!**

A1. (Grade: Y or N) Find the arc length of the curve given by

$x=\frac{1}{3}t^6+1$ and $y=\frac{1}{9}t^9-3$ for $0\le t\le 1$.

$$\frac{dx}{dt} = 2t^5 \qquad \frac{dy}{dt} = t^8$$

$$L = \int_0^1 \sqrt{(2t^5)^2+(t^8)^2}\, dt$$

$$= \int_0^1 \sqrt{4t^{10}+t^{16}}\, dt$$

$$= \int_0^1 t^5\sqrt{4+t^6}\, dt \qquad \begin{array}{l} u = 4+t^6 \\ du = 6t^5 dt \end{array}$$

$$= \frac{1}{6}\int_4^5 \sqrt{u}\, du = \frac{1}{6}\left(\frac{2}{3}u^{3/2}\right)\Big|_4^5$$

$$= \frac{1}{9}\left(5\sqrt{5}-8\right)$$

Answer A1: $\frac{1}{9}(5\sqrt{5}-8)$

B1. (Grade: Y or N) If the natural length of a spring is 2 meters and if it takes a force of 12 newtons to keep it extended to 6 meters, find the work done in stretching the spring from its natural length to a length of 3 meters.

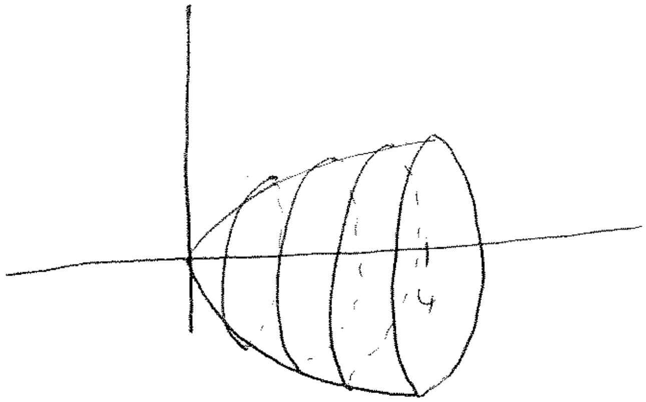
$12\,N = k(4\,m) \qquad k = 3\,N/m$

$W = \int_0^1 3x\,dx = \frac{3}{2}x^2\Big|_0^1$

$= \frac{3}{2}\,J$

Answer B1: $\boxed{\frac{3}{2}\text{ Joules}}$

C1. (Grade: Y or N) Find the surface area generated by revolving $y=3\sqrt{x}$ about the x-axis for $0 \le x \le 4$.



$$\frac{dy}{dx} = \frac{3}{2\sqrt{x}} \qquad \left(\frac{dy}{dx}\right)^2 = \frac{9}{4x}$$

$$SA = \int_0^4 2\pi \left(3\sqrt{x}\right) \sqrt{1 + \frac{9}{4x}} \, dx$$

$$= \int_0^4 6\pi\sqrt{x} \sqrt{\frac{4x+9}{4x}} \, dx$$

$$= 3\pi \int_0^4 \sqrt{4x+9} \, dx \qquad \begin{array}{l} u = 4x+9 \\ du = 4dx \end{array}$$

$$= \frac{3}{4}\pi \int_9^{25} \sqrt{u} \, du = \frac{\pi}{2} u^{3/2} \Big|_9^{25}$$

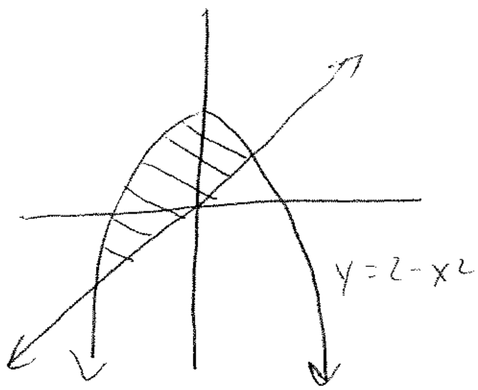
$$= \frac{\pi}{2}\left(125 - 27\right) = 49\pi$$

Surface Area = $\boxed{49\pi}$

D1. (Grade: Y or N) Find the centroid of the region bounded by $y=2-x^2$ and $y=x$.

$$2-x^2=x$$

$$x^2+x-2=0 \Rightarrow (x+2)(x-1)$$



$$M_y=\int_{-2}^{1} x[(2-x^2)-x]\,dx$$

$$=\int_{-2}^{1}(2x-x^3-x^2)\,dx$$

$$=x^2-\frac{x^4}{4}-\frac{x^3}{3}\Big|_{-2}^{1}$$

$$=\left(1-\frac{1}{4}-\frac{1}{3}\right)-\left(4-\frac{16}{4}+\frac{8}{3}\right)\Rightarrow$$

$$=\frac{5}{12}-\frac{8}{3}=\frac{5}{12}-\frac{32}{12}=-\frac{27}{12}=-\frac{9}{4}$$

$$m=\int_{-2}^{1}[(2-x^2)-x]\,dx=2x-\frac{x^3}{3}-\frac{x^2}{2}\Big|_{-2}^{1}$$

$$=\left(2-\frac{1}{3}-\frac{1}{2}\right)-\left(-4+\frac{8}{3}-2\right)$$

$$=8-3-\frac{1}{2}=\frac{9}{2}$$

$$M_x=\frac{1}{2}\int_{-2}^{1}((2-x^2)^2-x^2)\,dx=\frac{1}{2}\int_{-2}^{1}(4-5x^2+x^4)\,dx$$

$$=\frac{1}{2}\left(4x-\frac{5}{3}x^3+\frac{x^5}{5}\right)\Big|_{-2}^{1}=\frac{1}{2}\left(4-\frac{5}{3}+\frac{1}{5}\right)-\frac{1}{2}\left(-8+\frac{40}{3}-\frac{32}{5}\right)=\frac{9}{5}$$

$$\bar{x}=\frac{M_y}{m}=\frac{-\frac{9}{4}}{\frac{9}{2}}=-\frac{1}{2}$$

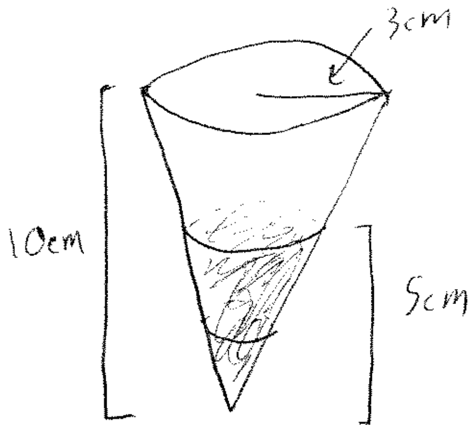
Answer D1: $\boxed{\left(-\frac{1}{2},\frac{2}{5}\right)}$

$$\bar{y}=\frac{M_x}{m}=\frac{\frac{9}{5}}{\frac{9}{2}}=\frac{2}{5}$$

Part 3: (20 points each) **Choose 5 out of the next 8 questions to do. Indicate clearly which problems you want graded!!!**

A2. (Grade: Y or N) A student is using a straw to drink from a conical paper cup, whose axis is vertical, at a rate of 3 cubic centimeters per second. If the height of the cup is 10 centimeters and the radius of its opening is 3 centimeters, how fast is the level of the liquid falling when the depth of the liquid is 5 cm?

(Note: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{h} = \frac{3}{10} \qquad r = \frac{3}{10}h$$

$$V = \frac{1}{3}\pi\left(\frac{3}{10}h\right)^2 h$$

$$= \frac{3\pi}{100}h^3$$

$$\frac{dV}{dt} = \frac{9\pi}{100}h^2\left(\frac{dh}{dt}\right) \qquad h = 5cm \qquad \frac{dV}{dt} = -3\ cm^3/s$$

$$-\frac{12\ cm^3/s}{9\pi} = \frac{dh}{dt} = -\frac{4}{3\pi}\ cm/s$$

Answer A2: $\boxed{\frac{dh}{dt} = -\frac{4}{3\pi}\ cm/s}$

B2. (Grade: Y or N) Solve the following differential equation

$$\frac{dy}{dx}=(x^2+3)\sec y \quad \text{such that} \quad x=3 \quad \text{when} \quad y=0 \ .$$

$$\frac{dy}{\sec y} = (x^2+3)\,dx$$

$$\Rightarrow \int \cos y \, dy = \int (x^2+3)\,dx$$

$$\sin y = \frac{x^3}{3} + 3x + C$$

$$\Rightarrow \sin(0) = 9 + 9 + C$$

$$\Rightarrow 0 = 18 + C \Rightarrow C = -18$$

$$\sin y = \frac{x^3}{3} + 3x - 18$$

$$\Rightarrow y = \sin^{-1}\left(\frac{x^3}{3} + 3x - 18\right)$$

$$\sin y = \frac{x^3}{3} + 3x - 18$$

or

Answer B2: $y = \sin^{-1}\left(\frac{x^3}{3} + 3x - 18\right)$

More room

C2. (Grade: Y or N) Use differentials to approximate the increase in volume of a chewing gum bubble when its radius increases from 200 mm to 203 mm.

$$V(r) = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\Delta V \approx 4\pi r^2 \Delta r = 4\pi (200mm)^2 (3mm)$$

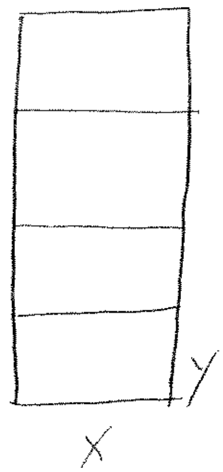
$$= 12\pi (40{,}000)\, mm^3$$

$$= 480{,}000\pi \; mm^3$$

Answer C2: $\boxed{480{,}000\pi \; mm^3}$

More room

D2. (Grade: Y or N) A farmer wishes to fence off four identical adjoining rectangular pens using exactly 400 feet of fence. What should the width and length of each pen be so that the area is the greatest?



$$P = 8y + 5x = 400$$

$$A = xy$$

$$y = 50 - \frac{5}{8}x$$

$$A = x\left(50 - \frac{5}{8}x\right) = 50x - \frac{5}{8}x^2$$

$$\frac{dA}{dx} = 50 - \frac{5}{4}x = 0 \Rightarrow \frac{200}{5} = x \Rightarrow x = 40$$

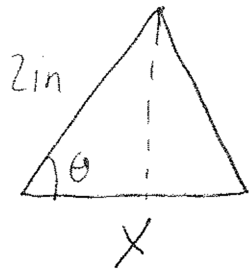
$$8y = 400 - 5(40) = 200$$

$$y = \frac{200}{8} = 25$$

Answer D2: 40 ft by 25 ft

Note: <u>Or</u> 25 ft by 40 ft

E2. (Grade: Y or N) For all isosceles triangles with congruent sides of length 2 inches, find the length of the base of the triangle with maximum area.



~~$2\sin\theta = h$~~

$$(2\,in)^2 = h^2 + \left(\frac{X}{2}\right)^2$$

$$h = \sqrt{4 - \frac{x^2}{4}}$$

$$A = \frac{1}{2} x \sqrt{4 - \frac{x^2}{4}}$$

$$\frac{dA}{dx} = \frac{1}{2}\left(x \left(\frac{-x/4}{\sqrt{4 - \frac{x^2}{4}}} \right) + \sqrt{4 - \frac{x^2}{4}} \right)$$

$$= \frac{1}{2}\left(\sqrt{4 - \frac{x^2}{4}} - \frac{x^2}{4\sqrt{4 - \frac{x^2}{4}}} \right) = 0$$

$$\Rightarrow 4\left(4 - \frac{x^2}{4}\right) = x^2$$

$$\Rightarrow 16 - x^2 = x^2 \Rightarrow 16 = 2x^2 \Rightarrow 8 = x^2$$

$$\Rightarrow x = 2\sqrt{2}$$

Answer E2: $\boxed{2\sqrt{2}\ in}$

F2. (Grade: Y or N) Use the **definition of the derivative** to find $f'(x)$ for $f(x)=\dfrac{1}{x^2-3}$.

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2-3} - \dfrac{1}{x^2-3}}{h}$$

$$= \lim_{h\to 0} \frac{(x^2-3) - [(x+h)^2-3]}{h[(x+h)^2-3](x^2-3)}$$

$$= \lim_{h\to 0} \frac{(x^2-3) - (x^2+2xh+h^2-3)}{h[(x+h)^2-3](x^2-3)}$$

$$= \lim_{h\to 0} \frac{-2xh-h^2}{h[(x+h)^2-3](x^2-3)} = \lim_{h\to 0} \frac{-2x-h}{[(x+h)^2-3](x^2-3)}$$

$$= \frac{-2x}{(x^2-3)^2}$$

Answer F2: $-\dfrac{2x}{(x^2-3)^2}$

G2. (Grade: Y or N) Use the definition of the definite integral, that is $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, to find $\int_0^3 (2x^2-1)\,dx$.

Δx = $\frac{3}{n}$

x_i = $\frac{3i}{n}$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(2\left(\frac{3i}{n}\right)^2 - 1\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{18i^2}{n^2} - 1\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{54i^2}{n^3} - \frac{3}{n}\right) = \frac{54}{n^3} \sum_{i=1}^{n} i^2 - \frac{3}{n} \sum_{i=1}^{n} 1$$

$$= \frac{54}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{3}{n}(n)$$

$$= \frac{9}{n^3}(2n^3+3n^2+n) - 3 = 18 + \frac{27}{n} + \frac{9}{n^2} - 3$$

$$= 15 + \frac{27}{n} + \frac{9}{n^2}$$

$$\lim_{n\to\infty}\left(15 + \frac{27}{n} + \frac{9}{n^2}\right) = 15$$

$\int_0^3 (2x^2-1)\,dx$ = $\boxed{15}$

H2. (Grade: Y or N) __ Find the value of c that satisfies the Mean Value Theorem for Integrals for $f(x)=\sqrt{2x-4}$ on [2, 4].

$$\int_2^4 \sqrt{2x-4}\,dx \qquad u = 2x-4$$

$$\tfrac{1}{2}du = dx$$

$$= \frac{1}{2}\int_0^4 \sqrt{u}\,du = \frac{1}{3}u^{3/2}\Big|_0^4$$

$$= \frac{8}{3}$$

$$\Rightarrow \frac{8}{3} = f(c)\,[4-2]$$

$$\Rightarrow \frac{4}{3} = \sqrt{2c-4}$$

$$\frac{16}{9} = 2c-4$$

$$\Rightarrow \frac{8}{9} = c-2$$

$$c = 2+\frac{8}{9} = \frac{26}{9}$$

c = $\boxed{\frac{26}{9}}$

More room

Extra Credit:

(10 points) Solve this equation using (A) the Bisection Method (just fill in table for first five rows) **OR** (B) Newton's Method (accurate to ~~four~~ eight decimal places). Choose which one you are doing, so we know what to grade.

$$f(x) = x^3 + 3x - 5 = 0 \quad \text{on } [0, 2]$$

Grade (A) or (B) *(circle one)*

(A) Bisection Method

n	a_n	b_n	m_n	$f(a_n)$	$f(b_n)$	$f(m_n)$
1	0	2	1	-5	9	-1
2	1	2	1.5	-1	9	2.875
3	1	1.5	1.25	-1	2.875	0.703125
4	1	1.25	1.125	-1	0.703125	-0.201171875
5	1.125	1.25	1.1875	-0.201171875	0.703125	0.2370605

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with $x_1 = 1$

(B) Newton's Method

$$f'(x) = 3x^2 + 3$$

$$x_{n+1} = x_2 - \frac{x_n^3 + 3x_n - 5}{3x_n^2 + 3}$$

n	x_n
1	1
2	$1.1\overline{6}$
3	1.154248366
4	1.154171498
5	1.154171495

$$\boxed{x \approx 1.15417149}$$