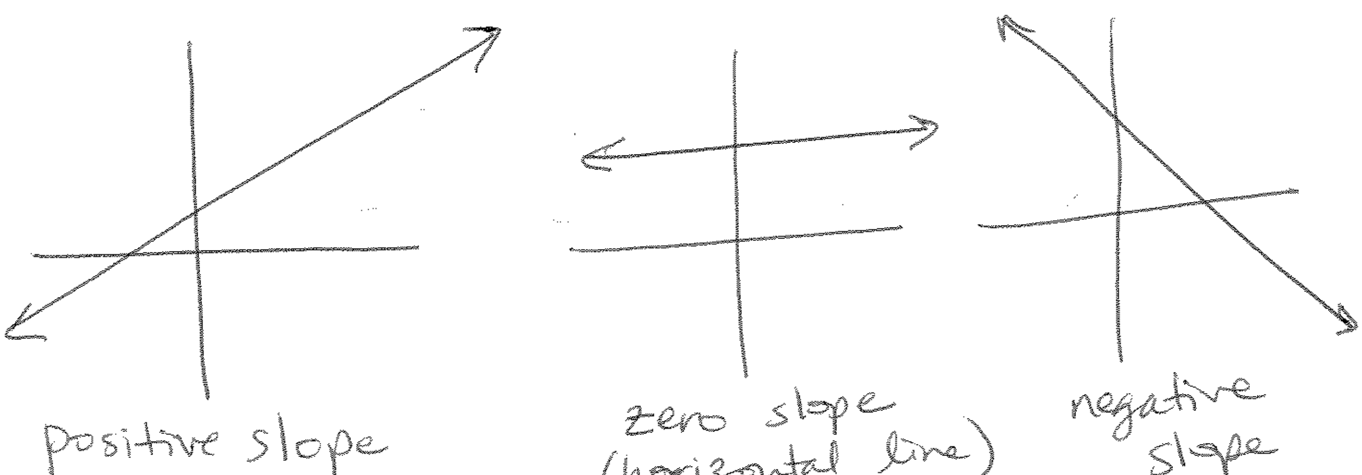
Math 1210

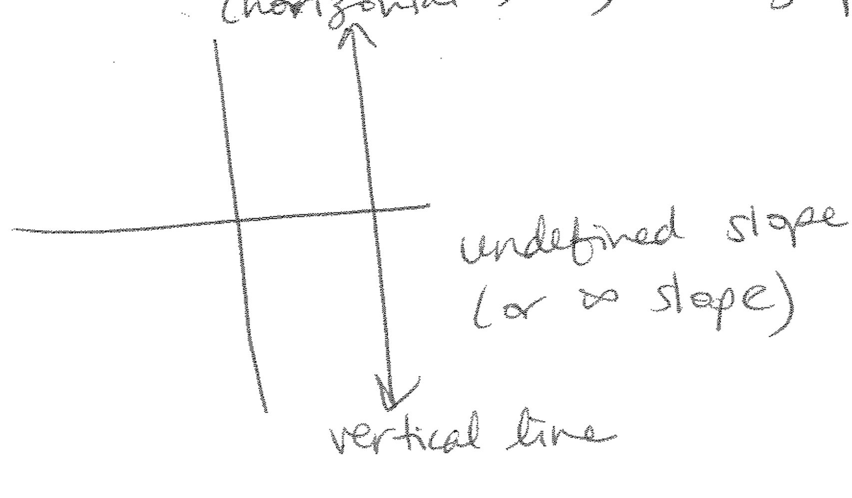
N1 (Polynomial Calculus)

- There is only one line between any 2 pts.
- slope of a line $\Rightarrow$ the steepness of the line; defined to be the vertical change over the horizontal change; denoted by m.

In a Cartesian coordinate system, if we have a line going through (x_1, y_1) and (x_2, y_2), then the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$





N1 (continued)

Ex Find the slope of the line that goes through $(-1,3)$ and $(2,4)$.

Point-Slope Form of a Line

Given $m =$ slope of my line and I know it goes through (x_1, y_1), then we know

$$m = \frac{y_1 - y}{x_1 - x}$$ for every point (x,y) on my line.

$$\Leftrightarrow \quad (x_1 - x)m = y_1 - y$$

$$\Leftrightarrow \quad (x - x_1)m = y - y_1$$

OR $\boxed{y - y_1 = m(x - x_1)}$

Slope-Intercept form of a Line

Given the slope m + the y-intercept $(0,b)$, the equation of the line is

$$y - b = m(x - 0) \Leftrightarrow \boxed{y = mx + b}$$

N1 (continued)

Ex Find the equation of the line going through $(-4, 1)$ and $(5, 2)$.

EX Find the equation of the line with slope $m = 3$ and y-intercept $(0, 5)$.

General Eqn of a Line

Every line can be written in the form

$$Ax + By + C = 0$$ where $A, B, C \in \mathbb{R}$.

A non-vertical line which can be written as

$$y = mx + b \iff -mx + y - b = 0$$, so it fits

the general form.

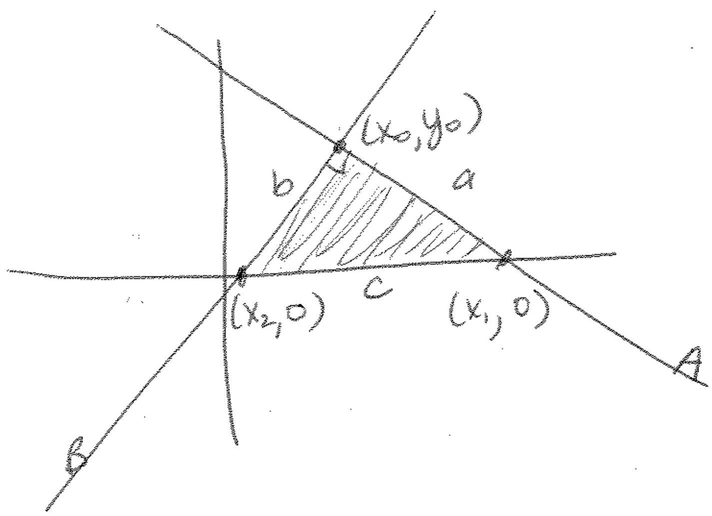
A vertical line is written as $x = k$ where k = constant.

N1 (continued)

Parallel and Perpendicular lines

① parallel lines have the same slope

② Perpendicular lines.



If lines A + B are perpendicular, then the shaded triangle is a right triangle. We can use Pythagorean Thm, $a^2+b^2=c^2$.

$$a^2=\left(\sqrt{(x_0-x_1)^2+(y_0-0)^2}\right)^2=(x_0-x_1)^2+y_0^2$$

$$b^2=\left(\sqrt{(x_0-x_2)^2+(y_0-0)^2}\right)^2=(x_0-x_2)^2+y_0^2$$

and $c^2=\left(\sqrt{(x_1-x_2)^2+(0-0)^2}\right)^2=(x_1-x_2)^2$

$$\Rightarrow a^2+b^2=(x_0-x_1)^2+y_0^2+(x_0-x_2)^2+y_0^2=(x_1-x_2)^2=c^2$$

$$\Leftrightarrow x_0^2-2x_0x_1+x_1^2+y_0^2+x_0^2-2x_0x_2+x_2^2+y_0^2$$
$$=x_1^2-2x_1x_2+x_2^2$$

N1 (continued)

$$\iff 2x_0^2 + \cancel{x_1^2} + \cancel{x_2^2} + 2y_0^2 - 2x_0x_1 - 2x_0x_2 = \cancel{x_1^2} - 2x_1x_2 + \cancel{x_2^2}$$

$$\iff 2x_0^2 + 2y_0^2 - 2x_0x_1 - 2x_0x_2 = -2x_1x_2$$

$$\iff x_0^2 + y_0^2 - x_0x_1 - x_0x_2 = -x_1x_2$$

$$\iff y_0^2 = -x_0^2 + x_0x_1 + x_0x_2 - x_1x_2$$

$$y_0^2 = -x_0(x_0 - x_1) + x_2(x_0 - x_1)$$

$$y_0^2 = (-x_0 + x_2)(x_0 - x_1)$$

$$y_0^2 = (x_2 - x_0)(x_0 - x_1)$$

$$\frac{y_0}{x_2 - x_0} = \frac{x_0 - x_1}{y_0}$$

and

m_A = slope of line A $= \dfrac{y_0 - 0}{x_0 - x_1} = \dfrac{y_0}{x_0 - x_1}$

m_B = slope of line B $= \dfrac{y_0 - 0}{x_0 - x_2} = \dfrac{y_0}{x_0 - x_2}$

$\Rightarrow \quad -m_B = \dfrac{1}{m_A}$ OR $\boxed{m_B = \dfrac{-1}{m_A}}$

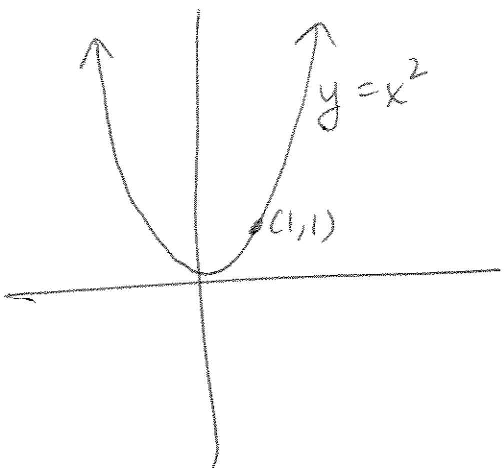
So, the slopes of $\perp$ lines are negative reciprocals of one another!

N1 (continued)

Ex Find the equation of the line perpendicular to $3x-4y=8$ and that passes through the point $(1, 3)$.

N2 Slope of a Curve

We know about slope of a line, but what about curves which don't have the same steepness everywhere?



Slope to left of origin:

Slope to right of origin:

Try to find the slope of the curve at the pt $(1,1)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - 1}{x_2 - 1}$$

2nd pt	slope
$(3, 9)$	
$(2, 4)$	
$(1.1, 1.21)$	
$(1.01, 1.0201)$	
$(1+h, (1+h)^2)$	

N2 (continued)

Ex Find the slope of the curve

$y = x^2 - 5x$ at $(2, -6)$.

Calculate slope between $(2, -6)$ and

$(2+h, (2+h)^2 - 5(2+h))$

N2 (continued)

Defn The slope of a function f at $(x, f(x))$ is given by $m = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f'(x)$ is called the derivative of f with respect to x.

other names for $f'(x)$ = slope
instantaneous rate of change
speed
velocity

Ex Find the derivative of $f(x) = 4x - 1$.

N3 Derivative of a Polynomial

Let's use our defn for a derivative to find the derivative of monomial functions.

$f(x)$	$f'(x)$
1	
x	
x^2	
x^3	
x^4	

For $f(x)=1$, $f'(x)=\lim_{h\to 0}\frac{1-1}{h}=0$

For $f(x)=x$,

$$f'(x)=\lim_{h\to 0}\frac{(x+h)-x}{h}=1$$

For $f(x)=x^2$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$=$$

For $f(x)=x^3$, $f'(x)=$

For $x^4=f(x)$, $f'(x)=$

N3 (continued)

What is our pattern?

Thm A If $n \in \mathbb{Z}^+$, then $(x^n)' = n x^{n-1}$.	The Power Rule

★ Other derivative notation $\dfrac{d(x^n)}{dx} = (x^n)'$.

Thm B The derivative of a constant times a function is that constant times the derivative of the function. And the derivative of a sum (or difference) of functions is the sum (or difference) of the derivatives of the functions.

i.e. $(cf)' = cf'$ and $(f+g)' = f' + g'$

Ex 1 Find the derivative of

$$y = -2x^2 + 5x - 1$$

Ex 2 Find the derivative of

$$f = 4x^8 - 3x^5 + 2x^2 + 7x$$

N3 (continued)

Ex 3 Find the slope of $y = 4x^3 - 3x^2 + 9x$ when $x = 1$.

Ex 4 Find the rate of change of $f(x) = -x^3 + 5x$ $\underbrace{\text{with respect to}}_{\text{(wrt)}}$ x when $x = -1$.

Velocity + Acceleration

velocity = how fast distance changes over time
= rate of change of distance wrt time
$= \frac{ds}{dt} = s'$ (if s = distance or position)

acceleration = how velocity changes over time
$= \frac{dv}{dt} = v'$

N3 (continued)

(#13) A ball is thrown straight up + follows the path given by

$$s(t) = -16t^2 + 32t + 6 \qquad t \text{ in seconds}$$


generic shape of curve

velocity = ? acceleration = ?
at time t

How high does the ball go?

N4 (Antiderivatives of Polynomials)

Defn If $f(x)$ is a function, then an antiderivative for f is a function having $f(x)$ as its derivative.

- An antiderivative "undoes" the derivative.
- For any function, there are infinitely many, or a "family," of antiderivatives.

We know $(x^{n+1})' = (n+1)x^n$

$\Leftrightarrow$

$\Leftrightarrow$

Defn B If $f(x)$ is a function, the set of all antiderivatives for $f(x)$ is denoted by $\int f(x)\,dx$ & is called the indefinite integral of $f(x)$.

$$\Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{where } C = \text{arbitrary constant}$$

$\forall\, n \in \mathbb{Z}^+$

Power Rule

N4 (continued)

Thm B Integral operator is linear, i.e.

(1) $\int a f(x)\,dx = a \int f(x)\,dx$

and

(2) $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

if f + g are functions and $a \in \mathbb{R}$.

Ex 1 $\int (x^2 - 4x + 7)\,dx$

Ex2 $F = \int 2x^3 - 5x^2\,dx = ?$ when we know $F = -4$ when $x = 0$.

N4 (continued)

Ex 3 The acceleration for an object due to gravity is $-32\ \text{ft/sec}^2$. A ball is thrown straight up with an initial velocity of 25 ft/sec, after which the only force acting on the ball is gravity. What is the velocity t seconds later? When does it reach its max ht?

$$a(t) = v'(t) \quad \Rightarrow$$

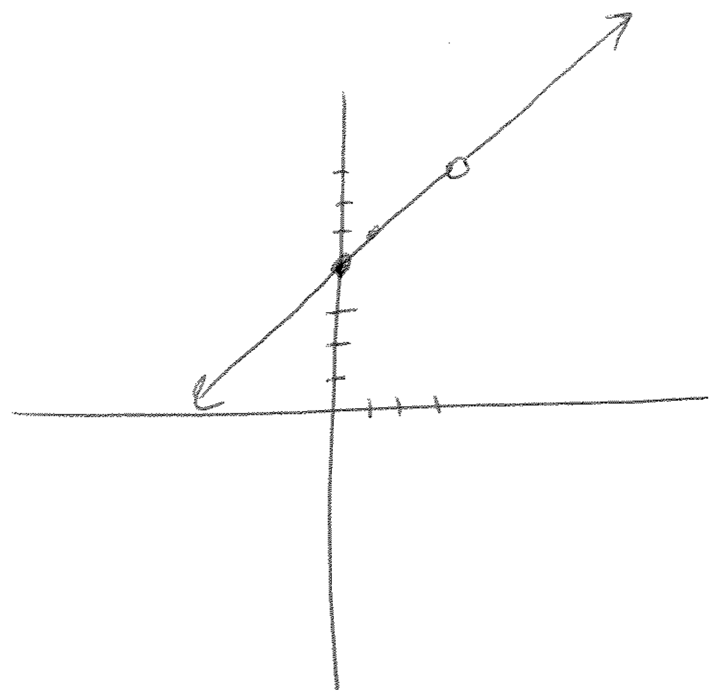
N4 (continued)

Ex 4 Refer to example 3, except this time the ball was thrown from an initial height of 6 ft. Find the height of the ball at any time t. What is the max ht?

1.1 Introduction to Limits

Consider $f(x) = \frac{x^2+x-12}{x-3}$. Note that it's undefined when $x=3$, since $f(3) = \frac{0}{0}$ which isn't a #. What happens as we approach $x=3$?

x	$f(x)$
3.25	7.25
3.2	7.2
3.1	7.1
3.05	7.05
3.01	7.01
3.001	7.001
⋮	
3	?
⋮	
2.99	6.99
2.95	6.95
2.9	6.9
2.8	6.8



So, as x approaches 3, it looks like $f(3) \to 7$.

We'd write $\lim_{x\to 3} \frac{x^2+x-12}{x-3} = 7$. We can compute this algebraically as

$$\lim_{x\to 3} \frac{x^2+x-12}{x-3} =$$

Defn To say $\lim_{x \to c} f(x) = L$ means that when x is near but different from c, then $f(x)$ is near L.

Ex 1 $\lim_{x \to 2} (3x+1)$

Ex 2 $\lim_{x \to 5} \dfrac{2x^2-7x-15}{x-5}$

Ex 3 $\lim_{x \to 9} \dfrac{x-9}{\sqrt{x}-3}$

1.1 (continued)

Ex 4 (This is a classic!)

$$\lim_{x\to 0} \frac{\sin x}{x}$$

Argument 1

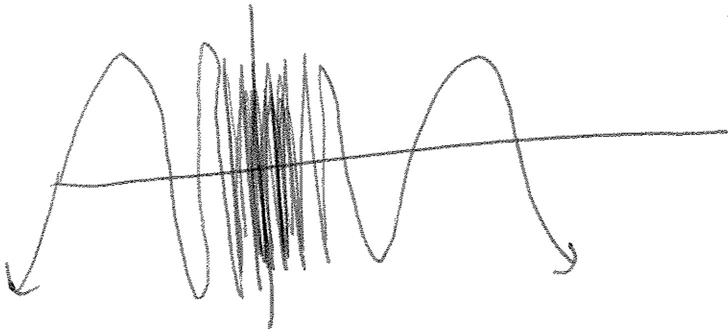
Use your calculator

x	$\frac{\sin x}{x}$
1.0	0.84147
0.5	0.95885
0.1	0.99833
0.01	0.99998
$\downarrow$	$\downarrow$
0	?
$\uparrow$	$\uparrow$
−0.01	0.99998
−0.1	0.99833
−0.5	0.95885
−1.0	0.84147

Argument 2

1.1 (continued)

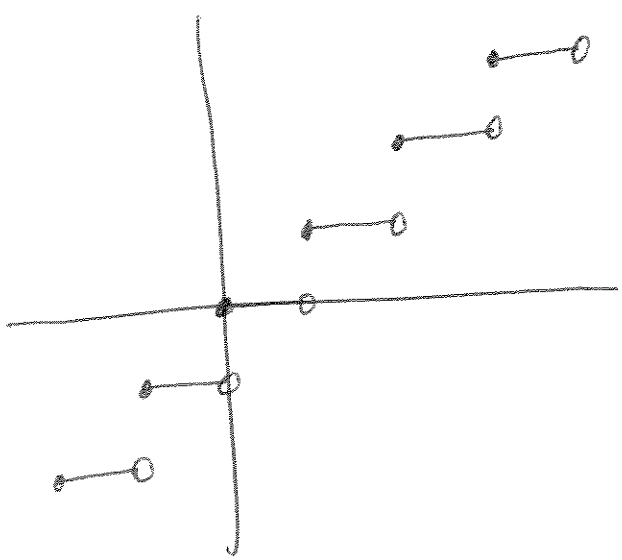
Ex 5 $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$



It wiggles way too much near $x=0$.

Ex 6 $\lim_{x \to 3} [\![x]\!]$

$[\![x]\!]$ = greatest integer function of; it returns the greatest integer $\leq x$.



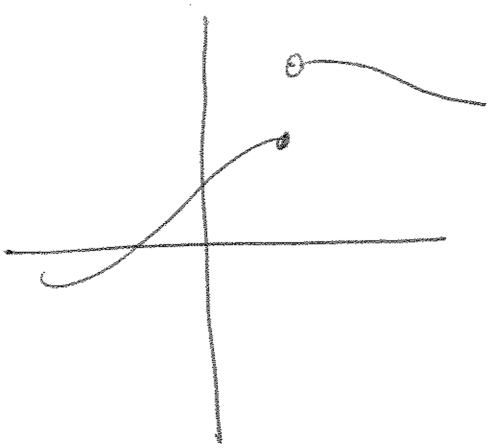
1.1 (continued)

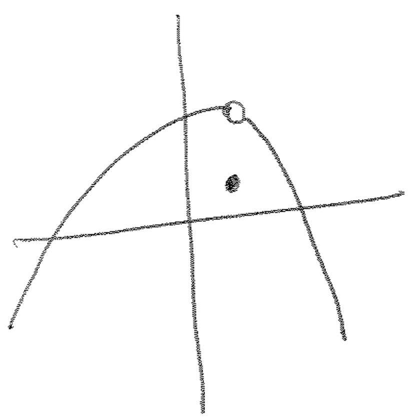
Defn Right + Left Hand Limits

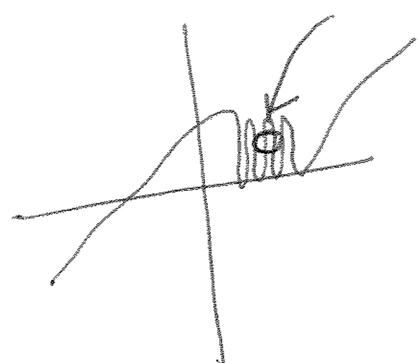
$\lim_{x \to c^+} f(x) = L$ means that when x approaches c from the right side of c, then $f(x)$ is near L.

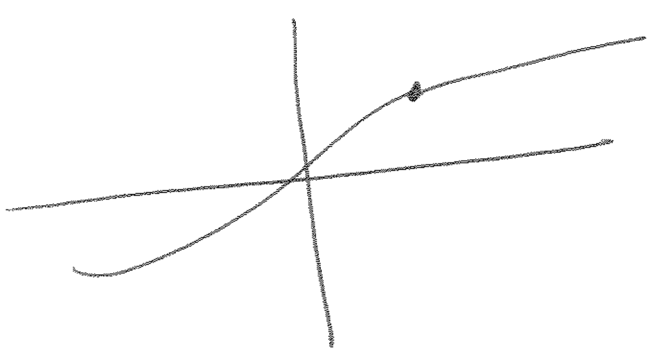
Likewise, $\lim_{x \to c^-} f(x) = L$ means that when x approaches c from the left side of c, then $f(x)$ is near L.

Thm A $L = \lim_{x \to c} f(x)$ iff $\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$









1.3 Limit Theorems

Main Limit Theorem Assume $n \in \mathbb{Z}^+$, $k \in \mathbb{R}$, and $f(x)$ + $g(x)$ have limits as $x \to c$.

(1) $\lim_{x \to c} k = k$

(2) $\lim_{x \to c} x = c$

(3) $\lim_{x \to c} k f(x) = k \lim_{x \to c} f(x)$

(4) $\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

(5) $\lim_{x \to c} [f(x) g(x)] = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$

(6) $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided $\lim_{x \to c} g(x) \neq 0$

(7) $\lim_{x \to c} [f(x)]^n = [\lim_{x \to c} f(x)]^n$

(8) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$ if $\lim_{x \to c} f(x) > 0$ $\forall$ n even

Ex 1 $\lim_{x \to 2} (4x^2 - 2x + 1)$

1.3 (continued)

Ex 2 $\lim_{x \to -3} \frac{\sqrt{x^2-1}}{2x}$

Ex 3 (#22) If $\lim_{x\to a} f(x) = 3$ + $\lim_{x\to a} g(x) = -1$,

find $\lim_{x\to a} \frac{2f(x)-3g(x)}{f(x)+g(x)}$.

1.3 (continued)

Substitution Thm

If f is a polynomial or rational function, then $\lim_{x \to c} f(x) = f(c)$ assuming $f(c)$ is defined.

Ex 4 $\lim_{x \to -1} \frac{3x^4 - 4x^3 + 7x - 5}{2x^2 + 3x + 4}$

Ex 5 $\lim_{x \to 2} \frac{3x^3 + 4x + 1}{x^2 - x - 2}$

1.3 (continued)

Squeeze Thm

Let $f, g + h$ be functns satisfying

$f(x) \le g(x) \le h(x) \quad \forall x$ near c, except possibly at $x=c$.

If $\lim_{x\to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

~~[illegible]~~

Ex 6 $\quad \lim_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$ (Hint: Rationalize numerator.)

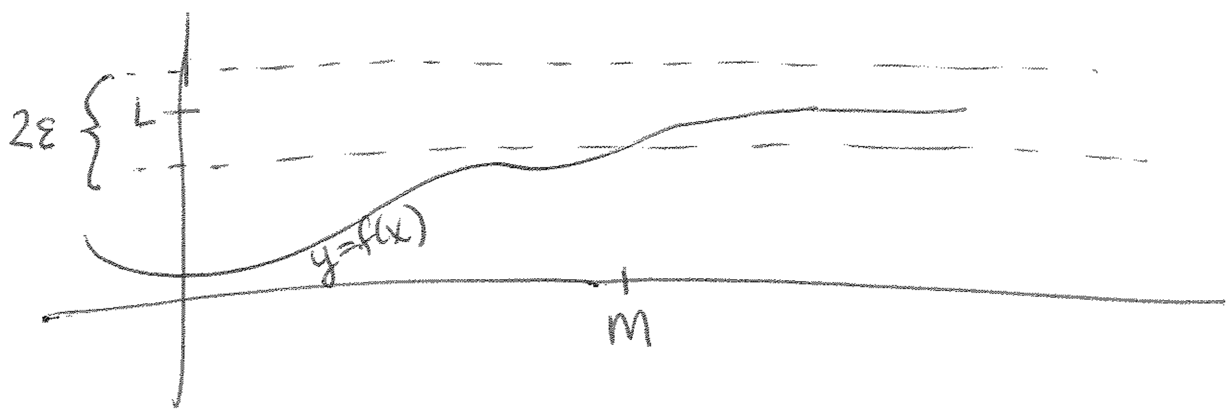
1.5 Limits at Infinity, Infinite Limits

Defn (limit as $x \to \pm\infty$)

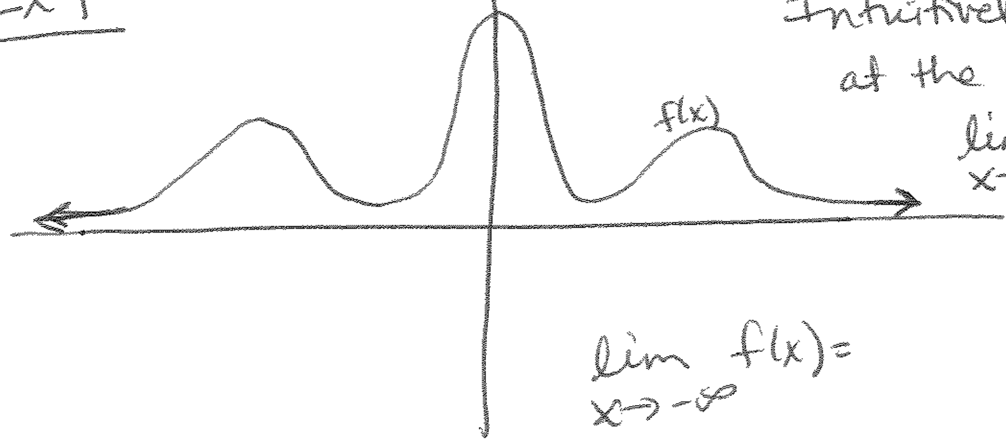
Let f be defined on $[c, \infty)$ (or $(-\infty, c]$) for some $c \in \mathbb{R}$. We say that $\lim_{x \to \infty} f(x) = L$ (or $\lim_{x \to -\infty} f(x) = L$) if $\forall\, \varepsilon > 0$ $\exists$ a corresponding number $M \ni$

$$x > M \text{ (or } x < M) \Rightarrow |f(x) - L| < \varepsilon$$

(M can be dependent on ε.)



Ex 1



Intuitively (looking at the graph) $\lim_{x \to \infty} f(x) = ?$

$\lim_{x \to -\infty} f(x) =$

1.5 (continued)

Ex 2 Show that if $n \in \mathbb{Z}^+$, then

$|x| > 1$ $\qquad \lim\limits_{x \to \infty} \frac{1}{x^n} = 0.$ $\quad$ (It's also true that $\lim\limits_{x \to -\infty} \frac{1}{x^n} = 0$)

Let $\varepsilon > 0$ be given. Choose $M = \left(\frac{1}{\varepsilon}\right)^{1/n} = \sqrt[n]{\frac{1}{\varepsilon}}$.

Then $x > M \Rightarrow \quad x^n > M^n \Rightarrow \frac{1}{M^n} > \frac{1}{x^n}$

So $\left|\frac{1}{x^n} - 0\right| = \left|\frac{1}{x^n}\right| < \frac{1}{M^n} = \frac{1}{\left[\left(\frac{1}{\varepsilon}\right)^{1/n}\right]^n} = \frac{1}{1/\varepsilon} = \varepsilon$

i.e. $\left|\frac{1}{x^n} - 0\right| < \varepsilon.$ $\qquad \Rightarrow$ by defn,

$$\lim_{x \to \infty} \frac{1}{x^n} = 0.$$

Ex 3 $\lim\limits_{x \to \infty} \frac{2x+3}{x^2+1} = \lim\limits_{x \to \infty} \left(\frac{2x+3}{x^2+1}\right)\left(\frac{1/x^2}{1/x^2}\right)$

1.5 (continued)

Ex 4 $\displaystyle\lim_{x\to\infty} \frac{3x^4-2x^3+5}{x^3+7}$

Ex 5 $\displaystyle\lim_{x\to\infty} \frac{2x^2+5x-1}{x^2+3x}$

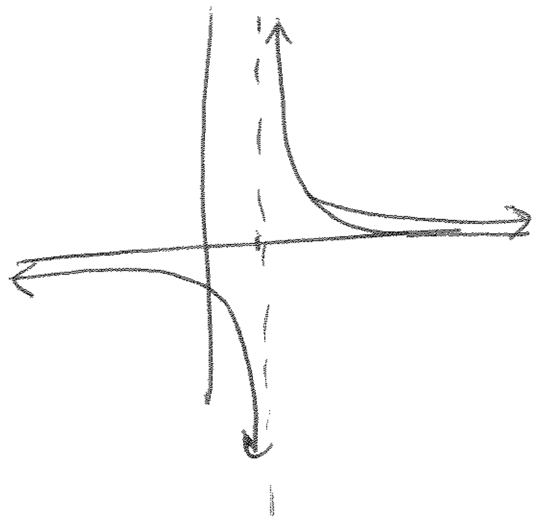
Defn (Infinite limit)

We say $\displaystyle\lim_{x\to c^+} f(x) = \infty$ if $\forall$ positive # M $\exists$ a corresponding $\delta > 0$ $\ni$

$$0 < x - c < \delta \Rightarrow f(x) > M.$$

i.e. If x is really close to c, $f(x)$ is bigger than some big # M. And, every time we get closer to c, $f(x)$ gets bigger.

1.5 (continued)



$f(x) = \frac{1}{x-1}$ has vertical asymptote $x=1$

You can see from graph,

$\lim_{x \to \infty} f(x) =$

$\lim_{x \to -\infty} f(x) =$

$\lim_{x \to 1^-} f(x) =$

$\lim_{x \to 1^+} f(x) =$

<u>Ex 6</u> Find horizontal & vertical asymptotes of $f(x) = \frac{-5x}{x-2}$

vertical asymptotes $\Rightarrow$ <u>always</u> come from restrictions on x

horizontal asymptotes $\Rightarrow$ describe height (y-value) of graph as x gets outrageously big (in a + or - direction)

vert. asymptote: $x=$

horiz. asymptote: $\lim_{x \to \infty} \frac{-5x}{x-2} =$

1.5 (continued)

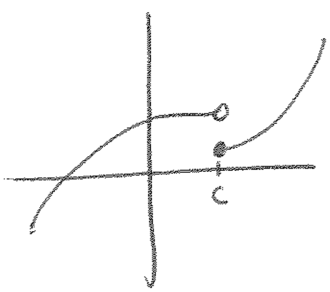
Ex 7 Find vert. + horiz. asymptotes

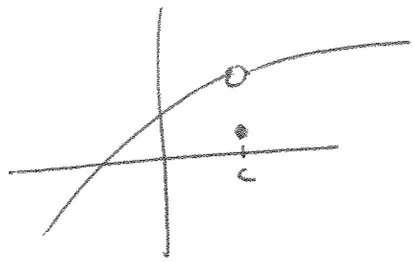
for $f(x) = \dfrac{2x}{\sqrt{x^2+5}}$

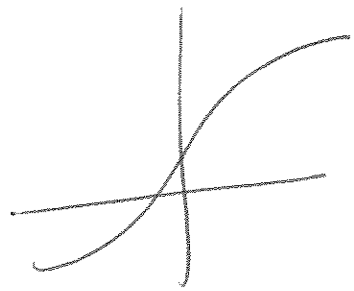
Ex 8 (We finally get to use the Squeeze Thm.)

$$\lim_{x \to \infty} x^{-1/2} \sin x$$

1.6 Continuity of Functions







Defn (Continuity at a Point)

Let f be defined on an open interval containing c. We say that f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

i.e. the functn needs to be ① defined at $x=c$, ② the limit needs to exist at $x=c$, and ③ the limit at $x=c$ needs to be exactly the function value.

Continuous Functions

a) All polynomials are continuous everywhere.
b) All rational functns are continuous $\forall x \in$ domain.
c) Absolute value function is continuous $\forall x \in \mathbb{R}$.
d) $f(x) = \sqrt[n]{x}$ continuous $\forall x \in \mathbb{R}$, if n odd.
e) $f(x) = \sqrt[n]{x}$ " $\forall x \in \mathbb{R}$ + non negative if n even.
f) sine + cosine fns are continuous $\forall x \in \mathbb{R}$.
g) $\cot x$, $\csc x$, $\sec x$, and $\tan x$ are continuous $\forall x \in$ domain

1.6 (continued)

More Continuous Fns

If f & g are continuous at c, then so are kf, $f+g$, $f-g$, fg, f/g (if $g(c)\neq 0$), f^n & $\sqrt[n]{f}$ (if $f(c)>0$ when n even).

Ex 1 State where these fns are continuous.

(a) $g(x) = x^2 - 9$

(b) $h(x) = \sqrt{x-5}$

(c) $f(x) = \dfrac{21-7x}{x-3}$

(d) $p(x) = \begin{cases} -3x+7 & x \leq 3 \\ -2 & x > 3 \end{cases}$

1.6 (continued)

Composite Limit Thm

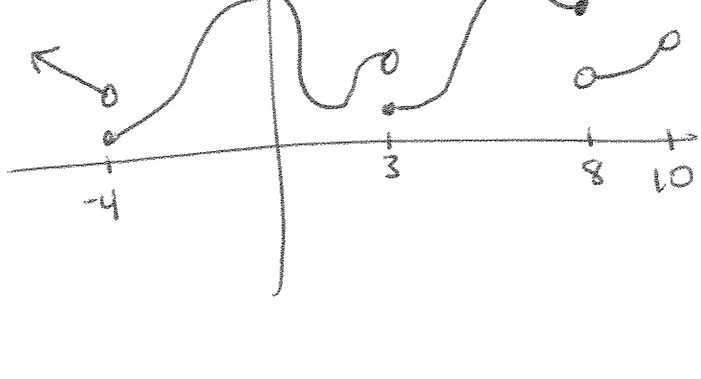
If $\lim_{x \to c} g(x) = L$ & f continuous at L, then
$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

Ex 2 At what pts are the following functions ~~[illegible]~~ continuous?

(a) $f(x) = \dfrac{1}{\sqrt{4+x^2}}$

(b) $g(t) = |t-2|$

Ex 3 Where is $f(x)$ continuous? (write answer in interval form)

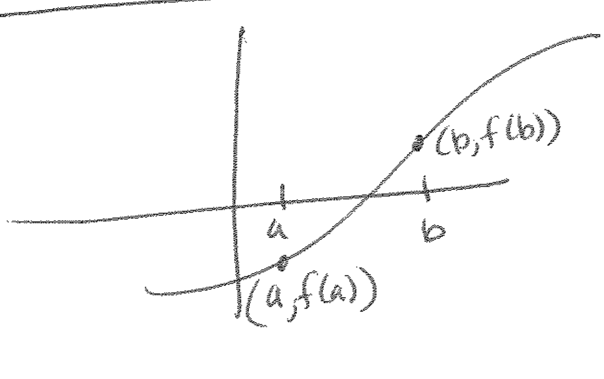


1.6 (continued)

Ex 4 If $f(x)=\dfrac{x^2-49}{x-7}$, how do we need to complete the defn for this fn to be continuous everywhere?

Intermediate Value Thm

f is a fn defined on $[a,b]$. W is # between $f(a)$ + $f(b)$. If f is continuous on $[a,b]$, then $\exists$ at least one # c between a + b $\ni$ $f(c)=W$.



Notes on Limits

<u>Setup: You have to find</u> $\lim_{x \to c} f(x)$

1. If c is a finite number, first try plugging in c.

   - If you get a finite number back, then you're done.

   - If you get the $\frac{0}{0}$ case, then you need to simplify more to find the limit. Keep going until you can plug in c for x.

   - If you get the $\frac{\textit{finite number}}{0}$ case, then it will either (1) go to ∞, (2) go to $-\infty$, or (3) it will not exist (DNE). You need to check the right and left-hand limits.
     - If the RH limit and the LH limit both go to ∞, then the limit also goes to ∞.

     - If the RH limit and the LH limit both go to $-\infty$, then the limit also goes to $-\infty$.

     - If the RH limit goes to $-\infty$ and the LH limit goes to ∞ (or the other way around), then the limit does not exist (DNE).

2. If c is $\pm\infty$ (and $f(x)$ is a rational function or at least has numerator and denominator that can be written as a collection of terms to powers), then

   - If the highest degree of the numerator > highest degree of the denominator, then the limit goes to $\pm\infty$. (You have to analyze the particular problem to decide if it's positive or negative infinity.)

   - If the highest degree of the numerator = highest degree of the denominator, then the limit is the quotient of the leading coefficients.

   - If the high degree of the numerator < highest degree of the denominator, then the limit is zero.

Asymptotes:

1. To find Vertical Asymptotes (VA)==>
   Look for x-values that will make the function undefined (e.g. x-values that make the denominator zero). Let's say that x = b makes function undefined.

   - If $\lim_{x \to b} f(x)$ equals a finite number, then there is only a "hole" (a.k.a. removable discontinuity) at x = b.

   - If $\lim_{x \to b} f(x)$ does not exist (or goes to $\pm\infty$), then x = b is a VA.

   * (A quick way to determine this is to try plugging in x = b in the function. If it goes to $\frac{0}{0}$, then it's a "hole." If it goes to $\frac{\text{finite number}}{0}$, then it's a VA.)

   A graph NEVER touches or crosses a vertical asymptote!!

2. To find Horizontal Asymptotes (HA)==>
   The HA can be found by calculating $y = \lim_{x \to \pm\infty} f(x)$. If the limit goes to $\pm\infty$, then there is no HA.

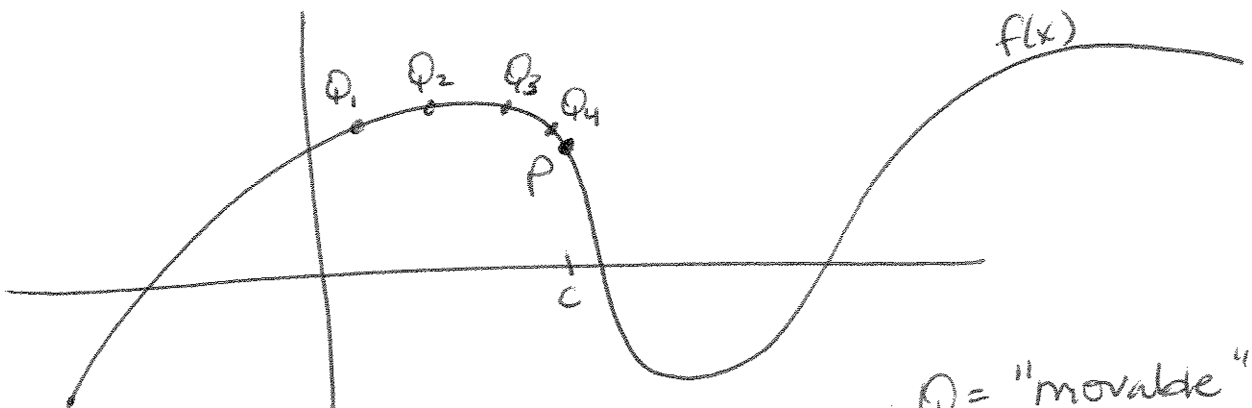
   *(If there is no HA and you have a rational function, you can always find the "slant asymptote" by doing long division.)

   Remember that the HA just describes the behavior of the graph as x gets really huge (either negatively or positively). A graph can cross and touch the horizontal/slant asymptotes as many times as it wants...but as x gets huge, it will only approach the asymptote.

2.1 Two Problems w/ One Theme

Archimedes – slope of a tangent line

Kepler/Galileo/Newton – instantaneous velocity



Q = "movable" pt

P = pt in question

<u>secant line</u> ⇒ line thru P + Q.

<u>tangent line</u> ⇒ limiting positn (if it exists) of secant line as Q moves twd P along the curve.

slope of secant line

$$m = \frac{f(c+h) - f(c)}{\cancel{c}+h-c} = \frac{f(c+h)-f(c)}{h}$$

slope of tangent line

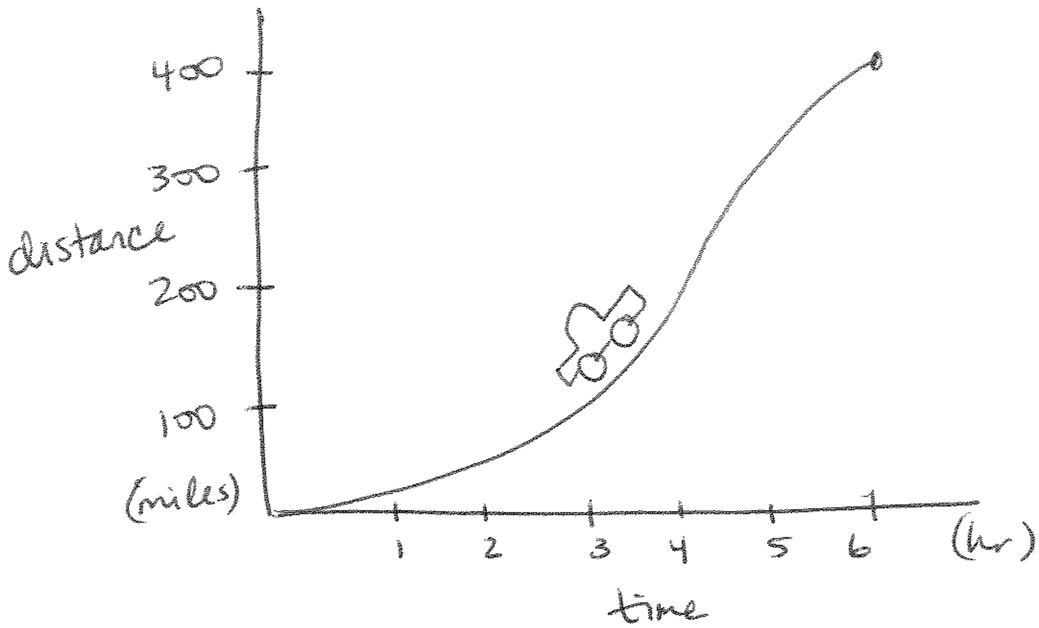
$$m = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$

2.1 (continued)

Ex 1 Find eqn of tangent line to $y = 2/x$ at $x = 1$.

Ex 2 Find slope of $y = -x^2 + 3x$ when $x = -1, 2$ & 5.

2.1 (continued)



t	d
3	120
2.5	75
2.1	54
2	50

If it takes me 6 hrs to drive 400 miles, then my avg. velocity is $\frac{400}{6} \simeq 67$ mi/hr.

But, surely I didn't drive that speed the whole time.

$V_{avg} = ?$ for different time intervals

start t	end t	V_{avg}
2	3	
2	2.5	
2	2.1	

$$V_{avg} = \frac{d_{end} - d_{start}}{t_{end} - t_{start}}$$

$\Rightarrow$ velocity at time $t = 2$ hrs $=$

2.1 (continued)

⇒ Geometrically finding slope of tangent line to a curve is exactly the same mathematical calculation as finding the instantaneous velocity for a moving object.

Ex 3 (#14) An object travels along a line so that its positn s is given by $s(t) = t^2 + 1$ (measured in meters, t measured in seconds).

(a) What is its avg velocity on interval $2 \le t \le 3$?

(b) Avg velocity on $2 \le t \le 2.003$?

(c) avg velocity on $2 \le t \le 2+h$?

(d) Instantaneous velocity at $t = 2$?

★ "rate of change" means instantaneous rate of change

2.2 The Derivative

Defn Derivative

The derivative of f is another function f'

$$\ni \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \forall x$$

provided the limit exists and is finite, for some x-value.

If $f'(c)$ exists, we say $f(x)$ is differentiable at $x = c$.

Ex 1 Find $f'(x)$ given $f(x) = 2\sqrt{x-1}$, $x \geq 1$

2.2 (continued)

Another form of defn of derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Ex2 Use above defn of f' to find $h'(c)$ if $h(x) = \dfrac{3}{x-5}$.

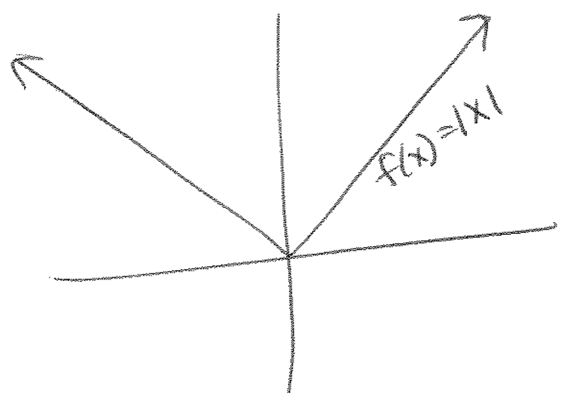
2.2 (continued)

Ex 3 Each of these is a derivative for some function. Can you find the function?

(a) $f'(x) = \lim_{h \to 0} \dfrac{3(x+h)^2 - 2(x+h) - (3x^2 - 2x)}{h}$

(b) $\lim_{x \to 3} \dfrac{\frac{4}{x} - \frac{4}{3}}{x-3}$

Ex 4 Let $f(x) = |x|$



Try to find $f'(0)$:

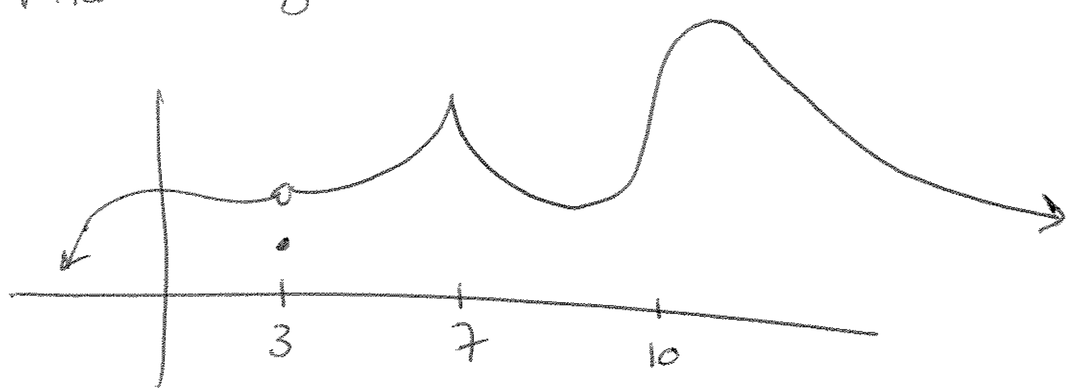
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} = \begin{cases} 1 & \text{if } h \to 0^+ \\ -1 & \text{if } h \to 0^- \end{cases}$$

$\Rightarrow$ $f'(0)$ DNE

2.2 (continued)

Visually, we can see a pt where the derivative (slope) DNE by looking for "corners" or vertical tangents in the graph of the function.



What can we say about derivative of this function at $x=3, 7$ and 10?

Thm Differentiability $\Rightarrow$ Continuity

If $f'(c)$ exists, then f is continuous at $x=c$.

Also if $f(c)$ is discontinuous $x=c$, then $f'(c)$ DNE.

2.3 Rules For Finding Derivatives

Thm **Constant Fn Rule**

If $f(x)=k$, $k\in\mathbb{R}$, $f'(x)=0$ (or $D_x(k)=0$).

Identity Fn Rule

If $f(x)=x$, then $f'(x)=1$ (or $D_x(x)=1$).

Power Rule

If $f(x)=x^n$, $n\in\mathbb{Z}^+$, $f'(x)=nx^{n-1}$ (or $D_x(x^n)=nx^{n-1}$).

D_x is linear operator

Constant Multiple Rule

If $k\in\mathbb{R}$, $f'(x)$ exists, then $D_x[kf(x)]=k(D_x(f(x)))$.

Sum + Difference Rule

If f' + g' exist, then

$$D_x[f(x)\pm g(x)] = D_x f(x) \pm D_x g(x)$$

Ex 1 Find $f'(x)$ if $f(x)=3x^7-4x^6+x^5+2x^3-x^2+4$

2.3 (continued)

Product Rule

If $f + g$ are differentiable, then

$$D_x[f(x)g(x)] = f(x)\, D_x[g(x)] + D_x[f(x)]\, g(x)$$

i.e. $(fg)' = f'g + g'f = g'f + f'g$

Ex 2 Find $f'(x)$ for $f(x) = (2x^3 - 4x + 1)(3x + 5)$

Use product rule:

Multiply out + use power rule to check:

2.3 (continued)

Quotient Rule

Let f + g be differentiable functions, $g(x) \neq 0$.

Then $$D_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\, D_x[f(x)] - f(x)\, D_x[g(x)]}{g^2(x)}$$

i.e. $$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

☆ I usually remember this one a bit differently.

If $f = \frac{\text{high}}{\text{low}}$, then $f' = \frac{\text{low} \cdot d(\text{high}) - \text{high} \cdot d(\text{low})}{\text{low}^2}$

"low d hi minus hi d lo over lo squared"

Ex 3 Find $f'(x)$ if $f(x) = \frac{2x^2+4x-1}{3x-2}$

2.3 (continued)

Ex 4 $y = \frac{-3}{x} + \frac{2}{x^4 - 7x}$ Find y'!

$$\Rightarrow \boxed{D_x(x^{-n}) = -n x^{-n-1}}$$ i.e. the Power Rule is true for -ve integers, too!

because

$f(x) = x^{-n} = \frac{1}{x^n}$ $\Rightarrow f'(x) = \frac{x^n(0) - 1(nx^{n-1})}{x^{2n}}$

$= \frac{-nx^{n-1}}{x^{2n}} = -nx^{n-1-2n}$

$= -nx^{-n-1}$

2.3 (continued)

Ex 5 Find $f'(x)$ if $f(x) = \frac{5x-4}{3x^2+1}$

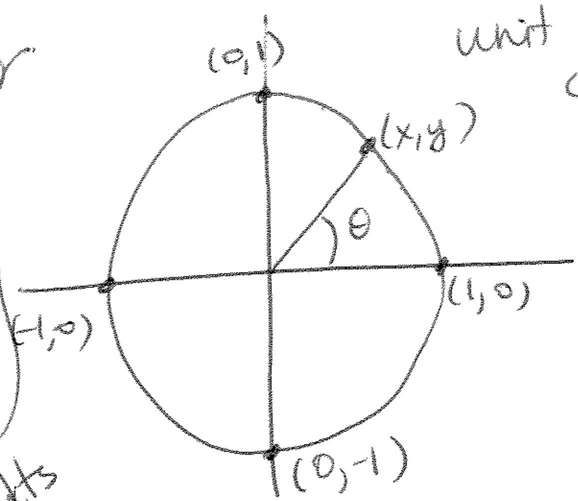
Ex 6 Find y' if $y = 3x(x^3-2x+1)$

Ex 7 Find $g'(x)$ if $g(x) = \frac{-3}{x^5} + \frac{2}{x}$

0.7 Trigonometric Fns

two contexts

(1) unit circle $(r=1)$ centered at $(0,0)$



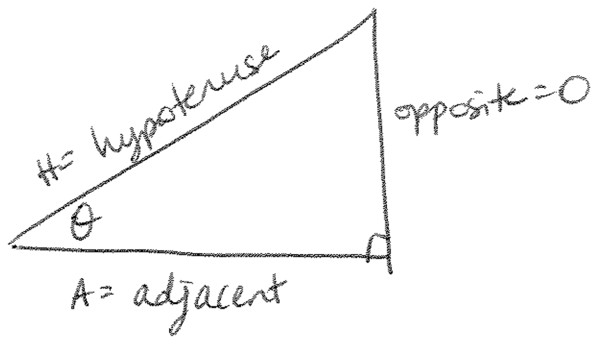
$\cos\theta = x$

$\sin\theta = y$

$-1 \le \sin\theta \le 1$

$-1 \le \cos\theta \le 1$

(2)



$\sin\theta = \dfrac{O}{H}$

$\cos\theta = \dfrac{A}{H}$

From Pythagorean Thm, we know

$A^2 + O^2 = H^2$ (for above triangle)

$\Leftrightarrow \dfrac{A^2}{H^2} + \dfrac{O^2}{H^2} = 1$ (divide both sides by H^2)

$\Leftrightarrow \left(\dfrac{A}{H}\right)^2 + \left(\dfrac{O}{H}\right)^2 = 1$

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

or we write it this way $\boxed{\cos^2\theta + \sin^2\theta = 1}$

0.7 (continued)

Other Trig Fns

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\qquad$ $\sec\theta = \dfrac{1}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ $\qquad$ $\csc\theta = \dfrac{1}{\sin\theta}$

If we look at $\sin^2\theta + \cos^2\theta = 1$, then

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \iff \boxed{1 + \cot^2\theta = \csc^2\theta}$$

Also $$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \iff \boxed{\tan^2\theta + 1 = \sec^2\theta}$$

Other Trig Properties

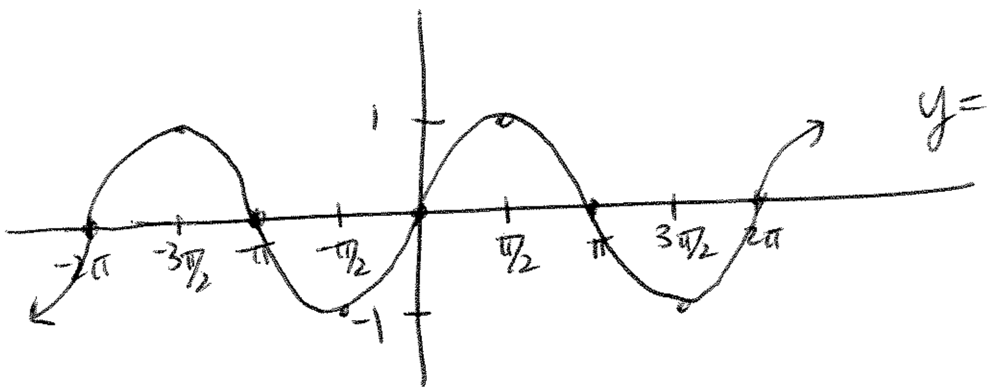
1. $\sin\theta = \sin(\theta + 2n\pi)$ $\quad \forall n \in \mathbb{Z}$
   & $\cos\theta = \cos(\theta + 2n\pi)$ $\quad \forall n \in \mathbb{Z}$

2. $\sin\theta$ is an odd fn, i.e. $-\sin\theta = \sin(-\theta)$
   & $\cos\theta$ is an even fn, i.e. $\cos\theta = \cos(-\theta)$
   (You can see this is true on the unit circle.)

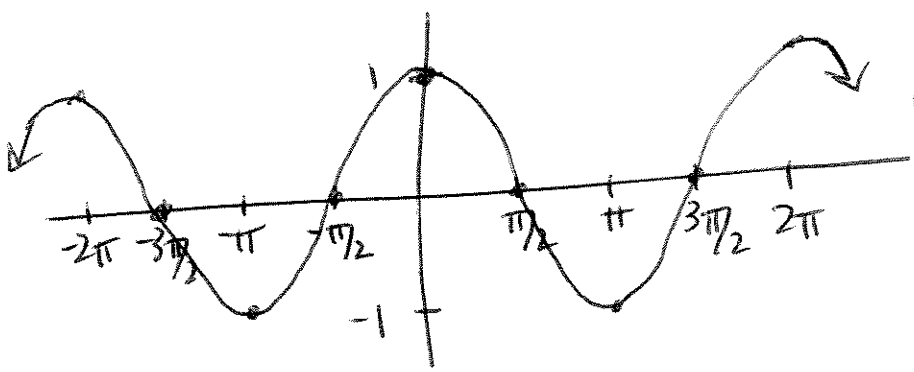
3. $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ $\quad$ &
   $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

0.7 (continued)

Graphs of $y = \sin x$ & $y = \cos x$



$y = \sin x$



$y = \cos x$

same shape – one graph is just shifted by $\pi/2$ horiz.

<u>amplitude</u> ⇒ half the distance between lowest to highest height of graph

<u>period</u> ⇒ the smallest # p ∋ $f(x+p) = f(x)$ for some fn $f(x)$.

For $\sin x$ & $\cos x$, the period is 2π, i.e. every 2π interval (on x-axis), the curve repeats itself.

0.7 (continued)

$f(x) = a\sin(b(x+c)) + d$ (likewise for $g(x) = a\cos(b(x+c))+d$)

amplitude

period of curve is $\frac{2\pi}{b}$

horizontal shift c units

vertical shift d units

$\boxed{180^\circ = \pi \text{ (radians)}}$ (★ Note Trig Properties in nice box on pg 47!)

Ex 1 (a) Convert $\frac{-\pi}{3}$ to degrees.

(b) Convert $\frac{3\pi}{18}$ to degrees.

(c) Convert -120° to radians.

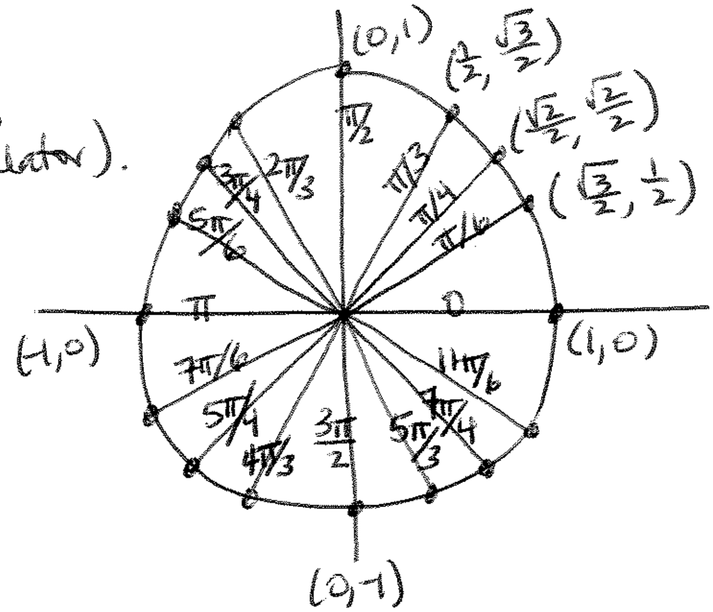
(d) Convert 600° to radians.

0.7 (continued)

Ex 2

Evaluate (w/o calculator).

(a) $\tan(\pi/6)$



(b) $\sec(\pi/3)$

(c) $\csc(\pi/4)$

(d) $\cos(-\pi/3)$

(e) $\cot(5\pi/6)$

(f) $\sin(5\pi/4)$

Ex 3 For the following fns, list the amplitude, period, horizontal + vertical shift + then graph.

(a) $y = 3\cos(x - \pi/2) - 1$

(b) $y = 2\sin(x + \pi/4)$

(c) $y = \frac{1}{2}\cos(2(x + \pi/2)) + 3$

0.7 (continued)

Ex 4 Verify the identity

$$\cos(3x) = 4\cos^3 x - 3\cos x$$

0.7 (continued)

Ex 5 (#30)

$\cos^2(\pi/12) = ?$

We know $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$

$\Rightarrow \cos^2\left(\frac{x}{2}\right) = \frac{1+\cos x}{2}$

$\cos^2(\pi/12) = \cos^2\left(\frac{\pi/6}{2}\right)$

$=$

1.4 Limits Involving Trig Fns

Thm $\forall c \in \mathbb{R}$ in the function's domain,

$$\lim_{x\to c} \sin x = \sin c \qquad \lim_{x\to c} \csc x = \csc c$$

$$\lim_{x\to c} \cos x = \cos c \qquad \lim_{x\to c} \sec x = \sec c$$

$$\lim_{x\to c} \tan x = \tan c \qquad \lim_{x\to c} \cot x = \cot c$$

i.e. Basically, we can still just plug in $x=c$, if it works.

Special Trig limits Thm

① $\displaystyle\lim_{t\to 0} \frac{\sin t}{t} = 1$ & ② $\displaystyle\lim_{t\to 0} \frac{1-\cos t}{t} = 0$

pf (of ②)

$$\lim_{t\to 0} \frac{1-\cos t}{t} = \lim_{t\to 0} \frac{(1-\cos t)}{t}\left(\frac{1+\cos t}{1+\cos t}\right)$$

$$= \lim_{t\to 0} \frac{1-\cos^2 t}{t(1+\cos t)} = \lim_{t\to 0} \frac{\sin^2 t}{t(1+\cos t)}$$

$$= \left[\lim_{t\to 0} \frac{\sin t}{t}\right]\left[\lim_{t\to 0} \frac{\sin t}{1+\cos t}\right]$$

$$= 1 \cdot 0 = 0$$

//

1.4 (continued)

Ex 1 $\lim_{x \to 0} \frac{3x \tan x}{\sin x}$

Ex 2 $\lim_{t \to 0} \frac{\sin^2 3t}{2t}$

Ex 3 $\lim_{\theta \to 0} \frac{\tan(5\theta)}{\sin(2\theta)}$

(Hint: $\sin 5\theta = \sin(3\theta + 2\theta) = \sin 3\theta \cos 2\theta + \cos 3\theta \sin 2\theta$
and $\sin 3\theta = \sin(\theta + 2\theta) = \sin\theta \cos 2\theta + \cos\theta \sin 2\theta$)

$$\lim_{\theta \to 0} \frac{\tan 5\theta}{\sin 2\theta} = \lim_{\theta \to 0} \frac{\sin 5\theta}{\sin 2\theta}\left(\frac{1}{\cos 5\theta}\right)$$

$=$

2.4 Derivatives of Trigonometric fns

Use the defn of the derivative to find $D_x(\sin x)$.

$$D_x(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right]$$

$$= \sin x \left[\lim_{h\to 0} \left(\frac{\cos h - 1}{h}\right)\right] + \cos x \left[\lim_{h\to 0} \frac{\sin h}{h}\right]$$

$$= -\sin x \left[\lim_{h\to 0} \frac{1-\cos h}{h}\right] + \cos x (1)$$

$$= -\sin x (0) + \cos x$$

$$= \cos x$$

We can follow the same argument to find $D_x(\cos x)$.

$\Rightarrow$ $\boxed{D_x(\sin x) = \cos x \quad \text{and} \quad D_x(\cos x) = -\sin x}$

Also

$$\boxed{\begin{array}{ll} D_x(\tan x) = \sec^2 x & D_x(\cot x) = -\csc^2 x \\ D_x(\sec x) = \sec x \tan x & D_x(\csc x) = -\csc x \cot x \end{array}}$$

2.4 (continued)

Ex 1 Find y' for the following fns.

(a) $y = \sin^2 x = (\sin x)(\sin x)$

(b) $y = \cot x$

(c) $y = \dfrac{x \cos x + \sin x}{x^2+1}$

(d) $y = \sin^2 x + \cos^2 x$

2.4 (continued)

Ex 2 Find the ~~eqn~~ equation of the tangent line to $y = \cot x$ at $x = \pi/4$.

2.5 The Chain Rule

Thm Chain Rule

$$D_x(f(g(x))) = f'(g(x))g'(x)$$

OR $D_x y = (D_u y)(D_x u)$

Basically, we differentiate from the "outside-in."

This is very useful if we need to differentiate something like $f(x) = 3(x^2-2x+1)^{80}$ + you really don't want to multiply it out 80 times.

Ex 1 If $y = (3x^3-4x+5)^{10}$, find y'.

Ex 2 $y = \dfrac{4}{(2x^7+6x^2)^5}$ find y'

2.5 (continued)

Ex 3 Find $f'(x)$.

(a) $f(x) = \sin^3(x^2-4x)$

(b) $f(x) = \left(\frac{2x+1}{x-5}\right)^4$

(c) $f(x) = \cos^2(\cos(\cos x))$

(d) $f(x) = \sin^2(4x)\,(2x^5+3x^2-1)^3$

2.5 (continued)

We can think of the chain rule as

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Ex 4 Find $\frac{dy}{dx}$.

(a) $y = [(2x^2+3)\cos x]^4$

(b) $y = \left(-3x + \frac{5}{x}\right)^{-4}$

(c)

2.6 Higher-Order Derivatives

Derivative	f' notation	y' notation	D_x notation	Leibniz notation
First	$f'(x)$	y'	$D_x y$	dy/dx
Second	$f''(x)$	y''	$D_x^2 y$	$\frac{d^2y}{dx^2}$
Third	$f'''(x)$	y'''	$D_x^3 y$	$\frac{d^3y}{dx^3}$
Fourth	$f^{(4)}(x)$	$y^{(4)}$	$D_x^4 y$	$\frac{d^4y}{dx^4}$
Fifth	$f^{(5)}(x)$	$y^{(5)}$	$D_x^5 y$	$\frac{d^5y}{dx^5}$
⋮ n^{th}	$f^{(n)}(x)$	$y^{(n)}$	$D_x^n y$	$\frac{d^ny}{dx^n}$

To get the second derivative, you need the first derivative first. It's a recursive process.

Ex 1 Find $f'''(x)$ for $f(x) = (3-5x)^5$

2.6 (continued)

Ex 2 Find $\frac{d^2y}{dx^2}$ for $y = x\sin\left(\frac{\pi}{x}\right)$

Ex 3 What is $D_x^5(3x^4 - 2x^3 + x^2 - 7)$?

Ex 4 Find a formula for $D_x^n\left(\frac{1}{x}\right)$.

2.6 (continued)

We know $a(t) = v'(t)$, i.e. acceleration = derivative of velocity wrt time.

$\Rightarrow$ $a(t) = v'(t) = s''(t)$

Ex 5 (# 24) An object moves along a horizontal coordinate line according to $s(t) = t^3 - 6t^2$. s = directed distance from origin (in ft.), t = time (in seconds).

(a) what are $v(t)$ + $a(t)$?

(b) when is the object moving to the right?

(c) when is it moving to the left?

2.6 (continued)

(d) When is its acceleration negative?

(e) Draw a schematic diagram that shows the motion of the object.

2.7 Implicit Differentiation

Given $2y^3 - y^2 = x^3 + 5$

This is a function y of x, i.e. $y(x)$, given implicitly (because we cannot solve directly for y in terms of x). So, how do we differentiate it? We start w/ the given eqn + differentiate both sides.

$$\frac{d}{dx}\left(2y^3 - y^2\right) = \frac{d}{dx}\left(x^3+5\right)$$

$$\Longleftrightarrow \quad \frac{d}{dx}(2y^3) - \frac{d}{dx}(y^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(5)$$

$$\Longleftrightarrow \quad 6y^2 y' - 2y\,y' = 3x^2$$

$$\Longleftrightarrow \quad y'\,(6y^2 - 2y) = 3x^2$$

$$\Longleftrightarrow \quad y' = \frac{3x^2}{6y^2 - 2y}$$

So, we now have the derivative $y' = \frac{dy}{dx}$ of our function. Notice that it is given in terms of x and y.

2.7 (continued)

Let's check to make sure implicit differentiation is reasonable.

If we have $x^2+2x^2y+3xy=0$, we can differentiate it two ways.

① Implicit $\quad 2x+4xy+2x^2y'+3y+3xy'=0$

$$2x^2y'+3xy'=-2x-4xy-3y$$

$$y'(2x^2+3x)=-2x-4xy-3y$$

$$y'=\frac{-2x-4xy-3y}{2x^2+3x}$$

② Explicit $\quad x^2+2x^2y+3xy=0$

$$\Leftrightarrow \quad y(2x^2+3x)=-x^2$$

$$\Leftrightarrow \quad y=\frac{-x^2}{2x^2+3x}$$

$$\Rightarrow y'=\frac{(2x^2+3x)(-2x)+x^2(4x+3)}{(2x^2+3x)^2}$$

Are they the same?

2.7 (continued)

Ex 1 Find $\frac{dy}{dx}$ for the following functions.

(a) $x\sqrt{y+1} = xy+1$

(b) $9x^2+4y^2=36$

2.7 (continued)

Ex2 Find the eqn of the tangent line at the indicated x-value.

(a) $y + \cos(xy^2) + 3x^2 = 4$ at $x=1$

(b) $\sqrt{y} + xy^2 = 5$ at $x=4$

2.7 (continued)

Power Rule (revisited)

Let $r \in \mathbb{Q}$. Then $\forall x > 0$
$$D_x(x^r) = r x^{r-1}.$$

Basically, the power rule now can be used with rational exponents

Why? Let $y = x^r = x^{p/q}$ $\quad p, q \in \mathbb{Z}, q \neq 0$

$$\Rightarrow \quad y^q = x^p$$

$$q y^{q-1} y' = p x^{p-1}$$

$$y' = \frac{p}{q}\left(\frac{x^{p-1}}{y^{q-1}}\right)$$

$$y' = \frac{p}{q}\left(\frac{x^{p-1}}{(x^{p/q})^{q-1}}\right)$$

$$y' = \frac{p}{q}\left(\frac{x^{p-1}}{x^{p-p/q}}\right)$$

$$y' = r\left(x^{p-1-p+p/q}\right)$$

$$y' = r\left(x^{p/q-1}\right)$$

$$y' = r x^{r-1}$$

//

Ex 3 Find y' if $y = \sqrt[3]{x} - 2x^{7/2}$

2.8 Related Rates

Remember that derivatives are rates. These problems are basically story problems that show the relevance of using derivatives. ☺

<u>Ex 1</u> Assuming that a soap bubble retains its spherical shape as it expands, how fast is its radius increasing when its radius is 3 inches if air is blown into it at a rate of 3 cubic inches per second?

$V = \frac{4}{3}\pi r^3$ volume of a sphere

We want to know $\frac{dr}{dt} = ?$ when $r = 3$ in.

And we're given that $\frac{dV}{dt} = 3 \frac{in^3}{sec}$.

$\Rightarrow$ Since $V = \frac{4}{3}\pi r^3$, then

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Leftrightarrow \quad \frac{dr}{dt} = \frac{dV}{dt}\left(\frac{1}{4\pi r^2}\right)$$

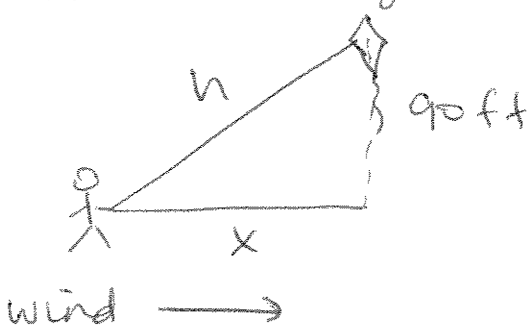
★ $\frac{dV}{dt}$ + $\frac{dr}{dt}$ are "related rates" here.

$\Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2}$ + when $r = 3$, $\frac{dr}{dt} = \frac{3}{4\pi(3^2)} = \frac{1}{12\pi}$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{12\pi} \simeq 0.0265 \frac{in}{sec}$$

2.8 (continued)

Ex 2 A child is flying a kite. If the kite is 90 ft above the child's hand level & the wind is blowing it on a horizontal course at 5 ft/sec, how fast is the child letting out the cord when 150 ft of cord is out? (Assume that the cord remains straight from hand to kite... not very realistic.)



$\frac{dx}{dt} = 5 \frac{ft}{sec}$

We want to know

$\frac{dh}{dt} = ?$ when $h = 150$ ft

2.8 (continued)

Ex 3 A student is using a straw to drink from a conical paper cup, whose axis is vertical, at a rate of 3 cm^3 per second. If the height of the cup is 10 cm, and the diameter of its opening is 6 cm, how fast is the level of the liquid falling when the depth of the liquid is 5 cm?

2.9 Differentials + Approximations

We've seen the notation $\frac{dy}{dx}$, and we're never separated the symbols. Now, we'll give meaning to dy + dx.

We know $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = f'(x_0)$ which

gives the derivative (slope) of the functn $f(x)$ at

$x = x_0$.

If Δx is really small, then

$$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \approx f'(x_0)$$

$$\Longleftrightarrow \quad \underbrace{f(x_0+\Delta x)-f(x_0)}_{\Delta y} \approx \underbrace{f'(x_0)\,\Delta x}_{dy}$$

Δy: actual change in y from x_0 to $x_0+\Delta x$

dy: an approximation of the change in y.

Defn Differentials

Let $y = f(x)$ be a differentiable function of x.

Δx is an arbitrary increment of x.

$dx = \Delta x$ (dx is called differential of x)

Δy is actual change in y as x goes from x to $x+\Delta x$

i.e. $\Delta y = f(x+\Delta x) - f(x)$.

$dy = f'(x)\,dx$ (dy is called differential of y)

2.9 (continued)

Ex 1 Find dy.

(a) $y = 4x^3 - 2x + 5$

$dy = (12x^2 - 2)\,dx$

(b) $y = 2\sqrt{x^4 + 6x}$

(c) $y = \cos(x^3 - 5x + 11)$

(d) $y = (x^{10} + \sqrt{\sin 2x})^2$

2.9 (continued)

Differentials can be used for approximations!

If $f(x+\Delta x) - f(x) \approx f'(x)\Delta x$, then

$$\boxed{f(x+\Delta x) \approx f(x) + f'(x)\Delta x}$$

Ex 2 Find a good approximation for $\sqrt{9.2}$ (w/o a calculator ☺).

Let $f(x) = \sqrt{x}$. when $x = 9$, $f(x) = 3$. Let $\Delta x = 0.2$.

$\Rightarrow f(x+\Delta x) \approx f(x) + f'(x)\Delta x$ becomes

$f(9.2) \approx f(9) + f'(9)(0.2)$ and we know

$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(9) = \frac{1}{2}(9^{-1/2}) = \frac{1}{2}\left(\frac{1}{\sqrt{9}}\right) = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}$

$\Rightarrow f(9.2) \approx \sqrt{9} + \frac{1}{6}(0.2) = 3 + \frac{0.1}{3} = 3 + 0.03\overline{3}$

$\Leftrightarrow f(9.2) \approx 3.0\overline{3}$ i.e. $\sqrt{9.2} \approx 3.0\overline{3}$

Ex 3 Use differentials to approximate the increase in the surface area of a soap bubble when its radius increases from 4 inches to 4.1 inches.

$A = 4\pi r^2$

Ex 4 The height of a cylinder is measured as 12 cm w/ a possible error of ± 0.1 cm. Evaluate the volume of the cylinder w/ radius 4 cm + give an estimate for the possible error in this value.

$V = \pi r^2 h = \pi (4)^2 h$

$\Rightarrow V = 16\pi h$

3.1 Maxima + Minima

Defn Let S, the domain of f, contain the point c.

Then (i) $f(c)$ is a max. value of f on S if $f(c) \geq f(x)\ \forall x \in S$.

(ii) $f(c)$ is a min. value of f on S if $f(c) \leq f(x)\ \forall x \in S$.

(iii) $f(c)$ is an extreme value of f on S if it is either a maximum or minimum value.

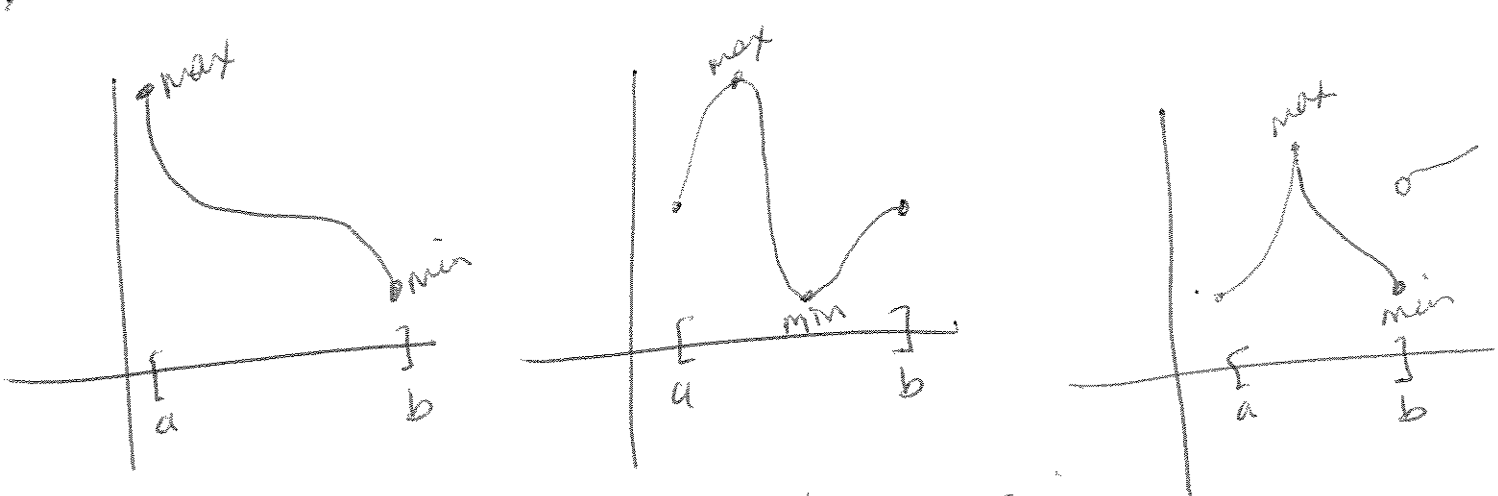
(iv) the function we want to maximize or minimize is called the objective function.

How do we know extreme values exist for a function?

Max-Min Existence Thm

If f is continuous on a closed interval $[a, b]$, then f attains both a max + min value on that interval.

☆ We need ① f continuous + ② a closed interval!



We can have max + min pts occur in one of 3 ways
① endpts of the closed interval
② stationary pts (where $f'(x)=0$)
③ singular pts (where derivative DNE)

3.1 (continued)

Critical pt Thm

Let f be defined on a closed interval I containing point c. If f(c) is an extreme value, then c is called a critical pt.

C is either ① an endpt of I.
or ② a stationary pt of f, i.e. $f'(c)=0$
or ③ a singular pt of f, i.e. $f'(c)$ DNE.

Ex 1 Find min & max values of

$$f(x) = -2x^3 + 3x^2 \quad \text{on } [-1, 3]$$

3.1 (continued)

Ex 2 Find the min & max points for
$f(x) = x^{2/5}$ on $[-1, 32]$.

Ex 3 Show that for a rectangle with perimeter of 30 inches, it has maximum area when it is a square.

3.1 (continued)

Ex 4 Identify critical pts, and specify min and max values.

$f(x) = x - 2\sin x$ on $[-2\pi, 2\pi]$

Ex 5 Sketch the graph of a function that is

① continuous, but not necessarily differentiable
② has domain $[0,6]$
③ reaches a max of 4 (at $x=4$)
④ reaches a min of 2 (at $x=2$)
⑤ and has no stationary pts.

3.2 Monotonicity + Concavity

Defn Let f be defined on an interval I (open, closed or neither). We say that:

① f is <u>increasing</u> on I if $\forall\ x_1 + x_2$ in I
$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

② f is <u>decreasing</u> on I if $\forall\ x_1 + x_2$ in I
$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

③ f is strictly <u>monotonic</u> on I if it is either increasing or decreasing on I.

Monotonicity Thm

Let f be continuous on I + differentiable at every interior pt of I.

① if $f'(x) > 0\ \forall\ x \in I$, then f is increasing on I.

② if $f'(x) < 0\ \forall\ x \in I$, then f is decreasing on I.

Ex 1 For $f(x) = x^3 + 3x^2 - 12$, find where f is increasing + decreasing.

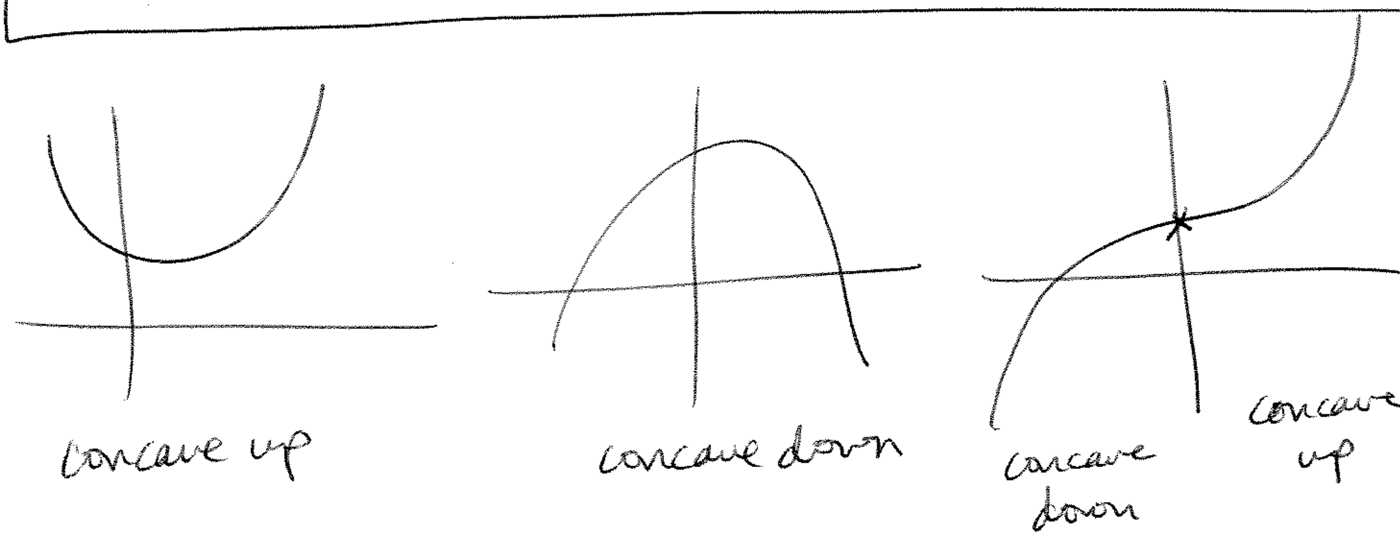
3.2 (continued)

Ex 2 Determine where $f(x) = \frac{x-1}{x^2}$ is increasing + decreasing.

Ex 3 Where is $f(x) = \cos^2 x$ $x \in [0, 2\pi]$ increasing + decreasing?

3.2 (continued)

Defn Let f be differentiable on an open interval I.
f is concave up on I if $f'(x)$ is increasing on I
+ f is concave down on I if $f'(x)$ is decreasing on I.



Concavity Thm

Let f be twice differentiable on an open interval I.
① If $f''(x) > 0$ $\forall x \in I$, f is concave up on I.
② If $f''(x) < 0$ $\forall x \in I$, f is concave down on I.

Ex4 Where is $f(x) = 4x^3 - 3x^2 - 6x + 12$ increasing, decreasing, concave up + concave down?

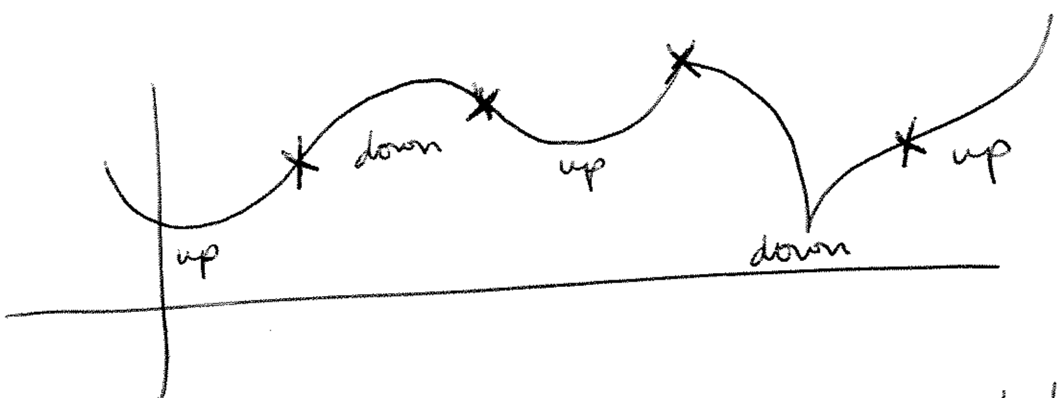
3.2 (Continued)

Ex 5 For $f(x) = 8x^{1/3} + x^{4/3}$, find where it's increasing, decreasing, concave up & concave down. Then, use this info to sketch the graph.

3.2 (continued)

Inflection Point

Let f be continuous at c. We call $(c, f(c))$ an inflection pt of f if f is concave up on one side of c + concave down on the other side of c.



X marks the inflectn pts.

We can find inflection pts by taking the second derivative. The x-values that make $f''(x) = 0$ or $f''(x)$ undefined are the <u>possible</u> x-values for the inflectn pts. You need to check out those possibilities.

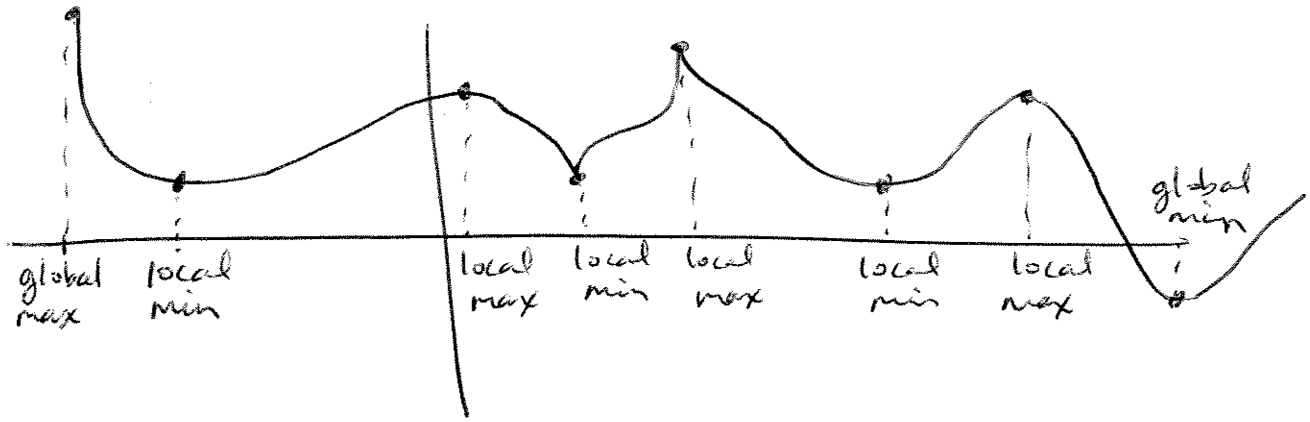
e.g. $\underline{f(x) = x^6}$

3.2 (continued)

Ex 6 Find all pts of inflection for

$$f(x) = 2x^{1/3} - 1$$

3.3 Local Maxima + Minima (and Extrema on Open Intervals)



Defn Let S = domain of f $\ni$ $c \in S$.

Then ① $f(c)$ is a local max value of f if $\exists$ (a,b) containing c $\ni$ $f(c)$ is max value ~~of~~ f on $(a,b) \cap S$.

② $f(c)$ is a local min value of f if $\exists$ (a,b) containing c $\ni$ $f(c)$ is a min value of f on $(a,b) \cap S$.

③ $f(c)$ is a local extreme value of f if it is either a local min or local max.

How do we find the local extrema?

First Derivative Test

Let f be continuous on an open interval (a,b) that contains a critical pt c.

① If $f'(x) > 0$ $\forall$ $x \in (a,c)$ + $f'(x) < 0$ $\forall$ $x \in (c,b)$, then $f(c)$ is a local max.

② If $f'(x) < 0$ $\forall$ $x \in (a,c)$ + $f'(x) > 0$ $\forall$ $x \in (c,b)$, then $f(c)$ is a local min.

③ If $f'(x)$ has same sign on both sides of c, then it's not a max nor a min.

3.3 (continued)

Ex 1 Find local min + max pts for $f(x) = 2x^2 - 5x + 3$.

Ex 2 Find local min + max pts for $f(x) = \frac{1}{2}x + \sin x$
$0 < x < 2\pi$

3.3 (continued)

Ex 3 Find all extreme values for $f(x) = x^4 + x^2 + 3$

Thm Second Derivative Test

Let f' + f'' exist at every pt in (a,b) containing c, and $f'(c) = 0$.

① If $f''(c) < 0$, $f(c)$ is local max.

② If $f''(c) > 0$, $f(c)$ is local min.

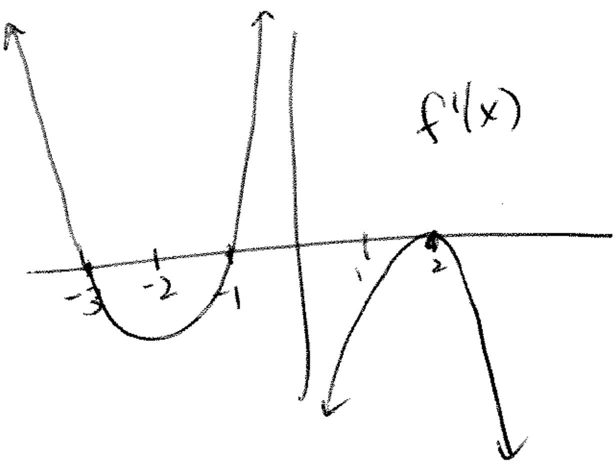
3.3 (continued)

Ex 4 Find all critical pts for $y = (x-2)^5$ + sketch the graph.

Ex 5 Find all min + max x-values for $y = x^2 + \frac{1}{x^2}$

3.3 (continued)

Ex 6 (#29) Let f be continuous + f' has the following graph.



Try to sketch a graph of $f(x)$ + answer these questions.

(a) where is f increasing? decreasing?

(b) where is f concave up? down?

(c) where does f attain a local max? min?

(d) where are the inflectn pts?

3.4 Practical Problems

<u>Ex 1</u> For what # does the principal square root exceed 8 times the # by the largest amt?

Let x= the #. Then we want to maximize $y = \sqrt{x} - 8x$.

<u>steps</u>

① Draw a picture or list info given.
② Write down what needs to be maximized or minimized.
③ If have more than 1 variable, find eqn to eliminate one.
④ Differentiate fn.
⑤ Set derivative $= 0$ (or find where it DNE) & solve. (i.e. find critical pts)
⑥ Check to make sure you found the max or min that you want.

3.4 (continued)

Ex 2 Find 2 #s whose product is −12 + sum of whose squares is a minimum.

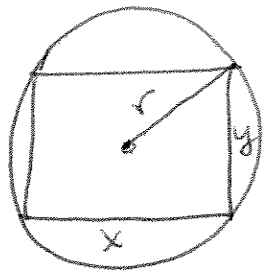
let x + y be the #s.

we know $xy = -12$ + we want to

minimize

3.4 (continued)

Ex 3 Show that the rectangle w/ max perimeter that can be inscribed in a circle is a square.



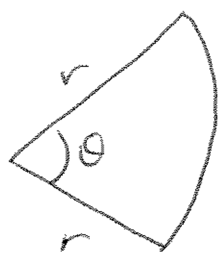
Let r be the radius of the circle + it's fixed.

What do we want to maximize?

What do we know (about relating x + y)?

3.4 (continued)

Ex 4 A flower bed will be in the shape of a sector of a circle (a pie-shaped region) of radius r & vertex angle θ. Find r & θ if its area is a constant A & the perimeter is a minimum.

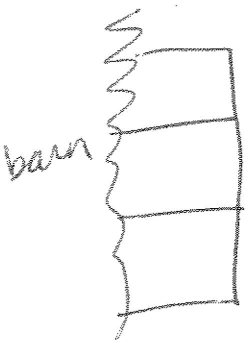


3.4 (continued)

Ex 5 (A true classic!) Find the volume of the largest open box that can be made from a piece of cardboard that is 24" by 9". You'll form the box by cutting out identical squares from the 4 corners + turning up the sides. Also, find the dimensions of the box that yields the max volume.

3.4 (continued)

Ex 6 A farmer has 80 ft of fence. He needs to enclose three identical pens along one side of his barn (the side along the barn needs no fence). What dimensions for the total enclosure make the area of the pens as large as possible?



3.5 Graphing Functions Using Calculus

Ex1 Sketch the graph of $f(x) = x^2(x^2-1)$.

(1) domain:

range:

(2) symmetry:

(3) x-intercepts:

(4) First Derivative info (increasing/decreasing):

(5) Second Derivative info (concavity/inflection pts):

(6) asymptotes:

(7) Sketch graph

3.5 (continued)

Ex 2 Sketch the graph of $f(x) = \frac{4x^4 - 8x^2 - 12}{3}$

3.5 (continued)

Ex 3 Sketch graph of $f(x) = \dfrac{(x-3)^2}{x}$

3.5 (continued)

Ex 4 Sketch the graph of $f(x) = |x|^3$

Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \Rightarrow \dfrac{d|x|}{dx} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

or we can say $\dfrac{d|x|}{dx} = \dfrac{x}{|x|}$ which covers it all.

3.6 The Mean Value Thm for Derivatives

Mean Value Thm for Derivatives

f continuous on $[a,b]$ + differentiable on (a,b).

Then $\exists$ at least one $c \in (a,b) \ni$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Leftrightarrow \quad f(b)-f(a) = f'(c)(b-a)$$

i.e. if $f(x)$ has nonvertical tangent line everywhere in (a,b), then there's at least one pt where the tangent line is parallel to the secant line connecting ~~[illegible]~~ the endpts.

Ex 1 Find the # c guaranteed by MVT for

$g(x) = (x+1)^3$ on $[-1,1]$.

Is $g(x)$ cont. on $[-1,1]$?

differentiable on $(-1,1)$?

3.6 (continued)

Ex 2 For $g(x) = \frac{x-4}{x-3}$, decide if we can use the MVT on (a) $[0, 5]$ or (b) $[4, 6]$. If so, use it to find the #c from the MVT. If not, state the reason why?

3.6 (continued)

Ex 3 For $f(x) = \csc x$ on $[-\pi, \pi]$, use MVT to find c.

Thm B

If $f'(x) = g'(x)$ $\forall\, x \in (a,b)$, then $\exists\; c \in \mathbb{R} \ni$

$$f(x) = g(x) + c \quad \forall\, x \in (a,b).$$

3.7 Solving Equations Numerically

(☆ you will definitely need a calculator for this section!)

Sometimes, we need to solve eqns of the form $f(x)=0$, to find the function's roots.

If it's too complicated to solve algebraically, we can use iterative steps numerically to close in on the solutions.

3 methods for solving eqns numerically

1. Bisection method
2. Newton's method
3. Fixed-Point method

1 Bisection method

Algorithm **Bisection Method**

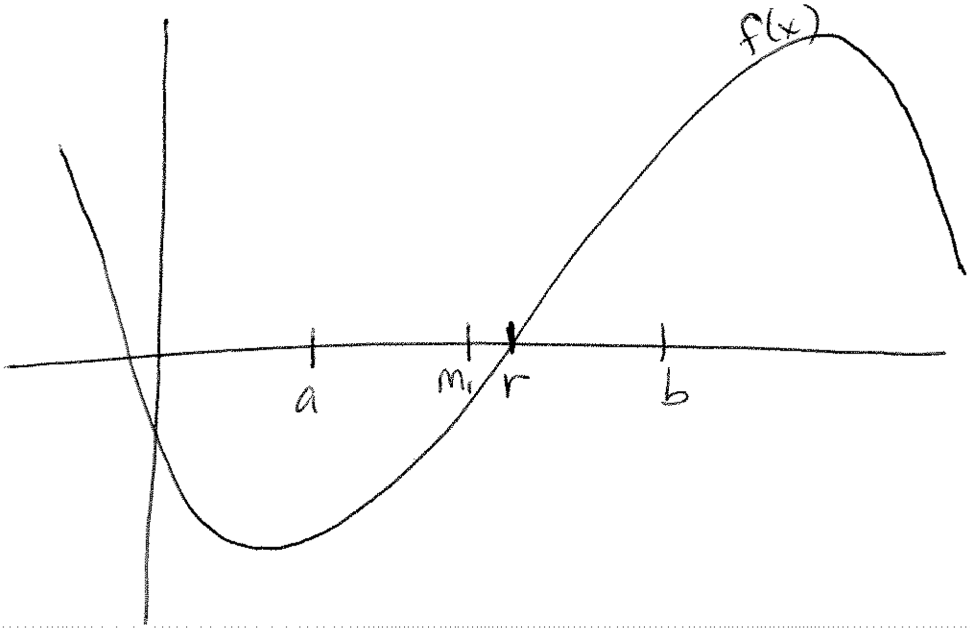
Let $f(x)$ be a continuous function, and let a_1 and b_1 be numbers satisfying $a_1 < b_1$ and $f(a_1) \cdot f(b_1) < 0$. Let E denote the desired bound for the error $|r - m_n|$. Repeat steps 1 to 5 for $n = 1, 2, \ldots$ until $h_n < E$:

1. Calculate $m_n = (a_n + b_n)/2$.
2. Calculate $f(m_n)$, and if $f(m_n) = 0$, then STOP.
3. Calculate $h_n = |(b_n - a_n)/2.|$ (for error testing)
4. If $f(a_n) \cdot f(m_n) < 0$, then set $a_{n+1} = a_n$ and $b_{n+1} = m_n$.
5. If $f(a_n) \cdot f(m_n) > 0$, then set $a_{n+1} = m_n$ and $b_{n+1} = b_n$.

⊕
- simple
- reliable

⊖
- can take a lot of time + computations (i.e. it's "computationally expensive")

3.7 (continued)



Basically, we're looking for $r \ni f(r)=0$. We know $f(a)<0$ + $f(b)>0$. (We choose a + b on either side of r.) Find the midpt between a + b (m_1) and find $f(m_1)$. If $f(m_1)=0$, then we're done. If not, then keep going, until you close in on r.

<u>Ex 1</u> Approximate real root of $f(x) = x^4 + 5x^3 + 1$ on $[-1, 0]$ to two decimal places.

(continued space on next page)

3.7 (continued)

Ex 1 (cont)

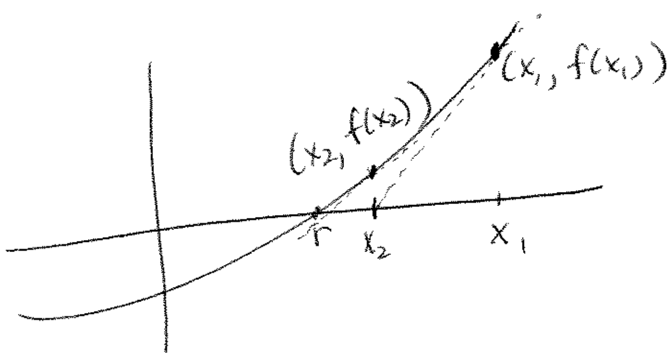
② Newton's method

Algorithm **Newton's Method**

Let $f(x)$ be a differentiable function and let x_1 be an initial approximation to the root r of $f(x) = 0$. Let E denote a bound for the error $|r - x_n|$.

Repeat the following step for $n = 1, 2, \dots$ until $|x_{n+1} - x_n| < E$:

1. $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



⊕
- finds root more quickly

⊖
doesn't always converge (you have to choose x_1 "close enough" to r)

3.7 (continued)

Basically, we choose x_1 + find tangent line to $f(x)$ at x_1. Find where that tangent line crosses x-axis and use that as x_2.
Repeat!

tangent line is $y - f(x_1) = f'(x_1)(x - x_1)$

This crosses x-axis when $y = 0$.

$$\Rightarrow \quad 0 - f(x_1) = f'(x_1)(x - x_1)$$

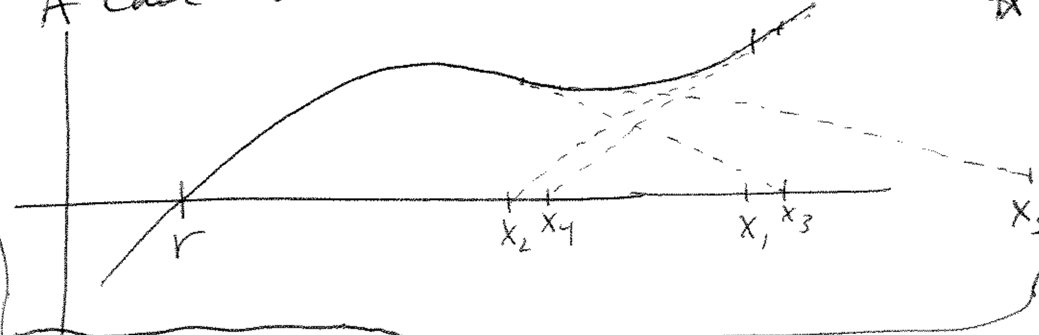
$$\frac{-f(x_1)}{f'(x_1)} = x - x_1$$

$$x_1 - \frac{f(x_1)}{f'(x_1)} = x \qquad \text{This is our } x_2!$$

$\Rightarrow$ Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Note: A case where Newton's method doesn't converge.



☆ you need to choose a closer start point

3.7 (continued)

Ex 2 Use Newton's method to approximate root of $7x^3+x-5=0$ to 5 decimal places

3.7 (continued)

(3) Fixed-Point Algorithm

Algorithm **Fixed-Point Algorithm**

Let $g(x)$ be a continuous function, and let x_1 be an initial approximation to the root r of $x = g(x)$. Let E denote a bound for the error $|r - x_n|$.

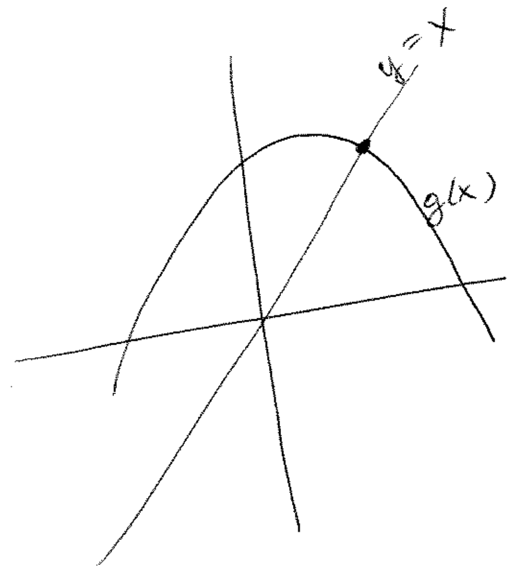
Repeat the following step for $n = 1, 2, \ldots$ until $|x_{n+1} - x_n| < E$:

1. $x_{n+1} = g(x_n)$

(+) simple

(−) Useful only when an eqn can be written in form $x = g(x)$.

Note: may or may not converge, depending on how close first guess is and form of $g(x)$ we choose.



Ex 3 Use the Fixed-Point Algorithm to solve $x = 2 - \sin x$, given $x_1 = 2$, to 5 decimal places:

3.8 Antiderivatives (Indefinite Integrals)

We already covered the gist of an antiderivative in section 0.4. Here are some reminders of what we did.

<u>Defn</u>
We call F <u>an</u> antiderivative of f on interval I, if $D_x F(x) = f(x)$ on I, i.e. if $F'(x) = f(x)\ \forall x \in I$.

<u>Power Rule Thm</u>
If $r \in \mathbb{R}$, except $r \neq -1$, then

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$$

<u>Indefinite Integral is a linear Operator</u>

Let f & g have antiderivatives & $k \in \mathbb{R}$. Then

(i) $\int k f(x)\,dx = k \int f(x)\,dx$

and (ii) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

<u>Ex 1</u> Evaluate the following integrals

(a) $\int (2x^4 + 3x^2 - 7)\,dx$

(b) $\int (u^3 - u^9)\,du$

3.8 (continued)

We know now that the power rule works for all rational exponents (besides -1).

Ex 2 Evaluate.

(a) $\int \left(\frac{1}{y^2} + y^{1/3}\right) dy$

(b) $\int \left(x^{-4} + \sqrt[3]{x^2} - \frac{3}{x^5}\right) dx$

3.8 (continued)

Thm

$$\int \sin x\, dx = -\cos x + C \quad \text{and} \quad \int \cos x\, dx = \sin x + C$$

Ex 3 $\int (t^2 - 2\cos t)\, dt$

Thm Generalized Power Rule

Let g be differentiable + $r \in \mathbb{Q}$, $r \neq -1$. Then

$$\int [g(x)]^r g'(x)\, dx = \frac{[g(x)]^{r+1}}{r+1} + C$$

Ex 4 $\int (4x^3+1)^4\, 12x^2\, dx$

3.8 (continued)

Ex 5 $\int (5x^2+1)\sqrt{5x^3+3x-2}\,dx$

Ex 6 $\int \frac{3y}{\sqrt{2y^2+5}}\,dy$

3.9 Intro to Differential Eqns

A differential eqn is an eqn that contains a derivative. We will need to integrate both sides, at some pt, to "undo" the derivative.

Ex 1 Find the eqn of the curve that goes through $(2,-4)$ + whose slope at any point on the curve is $3x$.

We know $\frac{dy}{dx} = 3x$. (First Order Separable D.E.)

$\Leftrightarrow \quad dy = 3x\,dx$

$\Leftrightarrow \quad \int dy = \int 3x\,dx$

$\Leftrightarrow \quad y = \frac{3}{2}x^2 + C$

We know it goes thru $(2,-4)$, so

$-4 = \frac{3}{2}(2^2) + C$

$-4 = 6 + C$

$C = -10$

$\Rightarrow \boxed{y = \frac{3}{2}x^2 - 10}$

3.9 (continued)

Ex 2 $\quad \dfrac{dy}{dx} = \sqrt{\dfrac{x}{y}} \qquad y = 4$ when $x = 1$

$$\frac{dy}{dx} = \frac{x^{1/2}}{y^{1/2}}$$

$$y^{1/2}\,dy = x^{1/2}\,dx$$

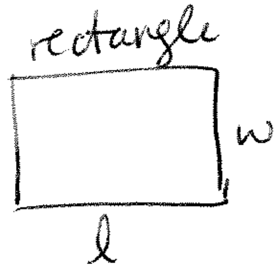
3.9 (continued)

Ex 3 $\frac{dy}{dx} = -y^2 x (x^2+2)^4$ thru $(0,1)$

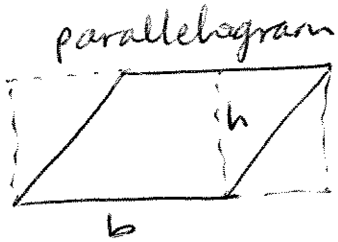
3.9 (continued)

Ex 4 The wolf population P in a certain state has been growing at a rate proportional to the cube root of the population size. The population was estimated at 1000 in 1980 + 1700 in 1990. When will the wolf population reach 4000?

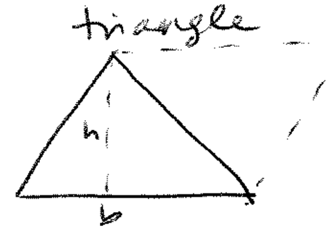
4.1 Introduction to Area



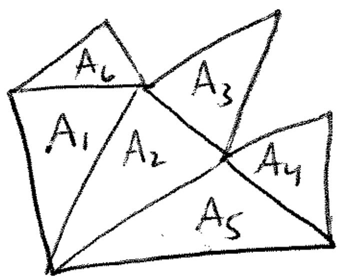
$A = lw$



$A = bh$

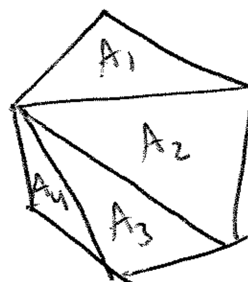


$A = \frac{1}{2}bh$

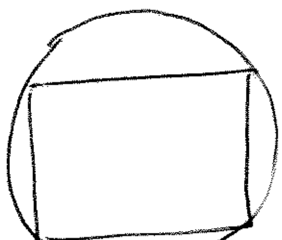


polygon

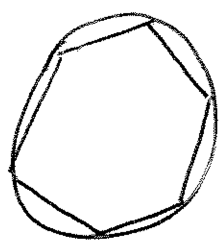
$A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$



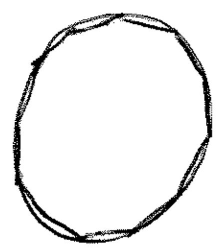
$A = A_1 + A_2 + A_3 + A_4$



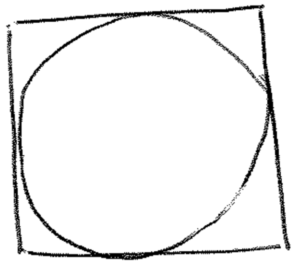
Estimate area of circle w/ inscribed rectangle.



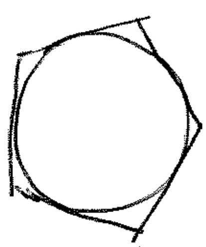
Estimate area better w/ hexagon



Estimate area of circle w/ inscribed polygon of increasing # of sides.



Estimate area of circle with circumscribed square.



Estimate area w/ pentagon.

Estimate area of circle w/ circumscribed n-gon.

4.1 (Sums + Sigma Notation)

$$1+2+3+4+\dots+100 = \sum_{i=1}^{100} i$$

OR $$2+4+6+8+\dots+1000 = \sum_{i=1}^{500} 2i$$

OR $$1+4+9+16+\dots+625 = \sum_{i=1}^{25} i^2$$

sequence

$\{a_i\}$ notation

$a_1, a_2, a_3, \dots$

e.g.

$1, 1, 2, 3, 5, 8, \dots$

$\sum$ = capital Greek symbol called "sigma"; it means summation

i = index (we can also call this j or k, or whatever)

$$\sum_{j=1}^{n} \frac{1}{j} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$$

$$\boxed{\sum_{i=1}^{n} c = \underbrace{c+c+c+\dots+c}_{n \text{ times}} = nc}$$

where c is independent of index

Linearity of $\sum$

Let $\{a_i\}$ + $\{b_i\}$ denote 2 sequences + $c \in \mathbb{R}$. Then,

(i) $\displaystyle\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

+ (ii) $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

4.1 (continued)

Special Sum Formulas *(see proofs pg 225) for ① + ②

① $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

② $\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

③ $\sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

④ $\sum_{i=1}^{n} i^4 = 1^4+2^4+3^4+\cdots+n^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$

Convincing argument from Gauss.

4.1 (Continued)

$$\boxed{\underline{\text{Collapsing Sum}} \quad \sum_{i=1}^{n}(a_{i+1}-a_i) = a_{n+1}-a_1}$$

<u>Ex 1</u> $\displaystyle\sum_{k=1}^{10}(2^k - 2^{k-1})$

<u>Ex 2</u> $\displaystyle\sum_{k=3}^{m+1}(a_k - a_{k-1})$

4.1 (continued)

Ex 3 $\displaystyle\sum_{i=1}^{10} [(i-1)(4i+3)]$

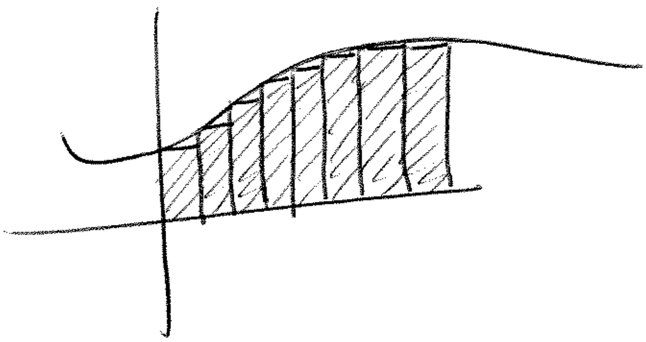
Ex 4 $\displaystyle\sum_{i=1}^{7} (2i-3)^2$

Ex 5 (#32) change the variable in the index to start at 1.

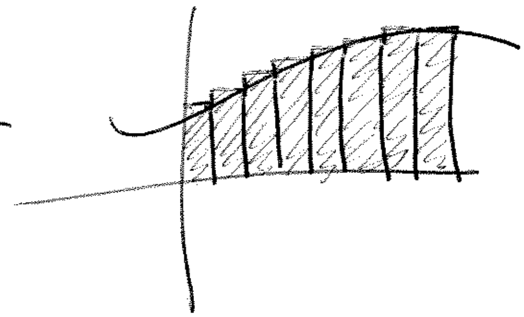
$\displaystyle\sum_{k=5}^{14} k\,2^{k-4}$

4.1 (continued)

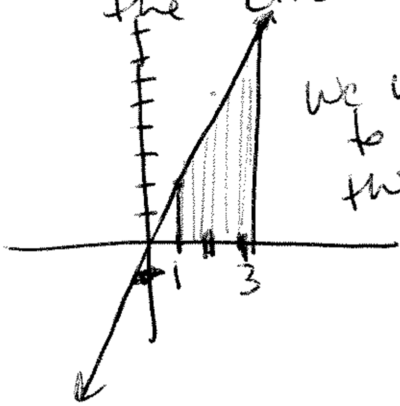
To find area under curve, we can estimate it w/ inscribed or circumscribed rectangles.



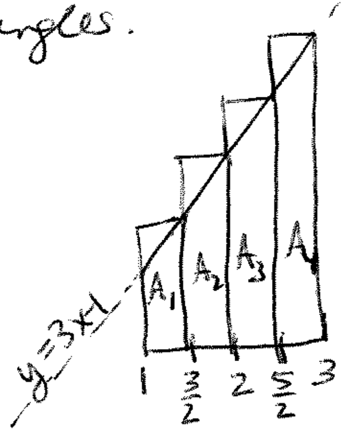
OR



Ex 6 For $f(x) = 3x-1$, divide the interval $[1, 3]$ into 4 equal subintervals. Calculate area of the circumscribed rectangles.



We want to estimate this area.



Here's our estimation.

$$A = A_1 + A_2 + A_3 + A_4$$

$$= \left(\tfrac{1}{2}\right)\left(f\left(\tfrac{3}{2}\right)\right) + \left(\tfrac{1}{2}\right)\left(f(2)\right) + \tfrac{1}{2}\left(f\left(\tfrac{5}{2}\right)\right) + \tfrac{1}{2}\left(f(3)\right)$$

base * ht + base * ht + base * ht + base * ht

$$= \tfrac{1}{2}\left(3\left(\tfrac{3}{2}\right)-1\right) + \tfrac{1}{2}\left(3(2)-1\right) + \tfrac{1}{2}\left(3\left(\tfrac{5}{2}\right)-1\right) + \tfrac{1}{2}\left(3(3)-1\right)$$

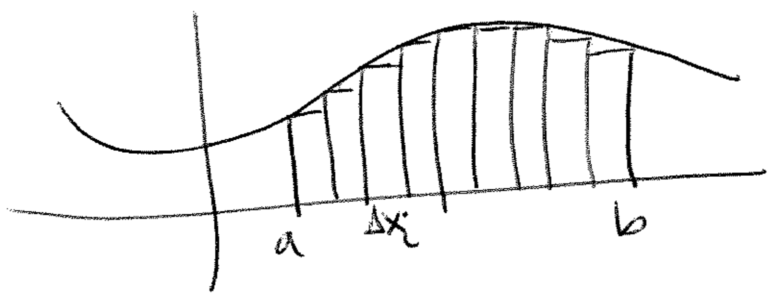
$$= \tfrac{1}{2}\left(\tfrac{7}{2}\right) + \tfrac{1}{2}(5) + \tfrac{1}{2}\left(\tfrac{13}{2}\right) + \tfrac{1}{2}(8)$$

$$= \tfrac{7}{4} + \tfrac{10}{4} + \tfrac{13}{4} + \tfrac{16}{4}$$

$$= \tfrac{46}{4} = \tfrac{23}{2}$$

4.2 The Definite Integral

In general, we can find the area under a curve using inscribed or circumscribed rectangles.



The area of each rectangle is base * ht, where base $= \Delta x_i$ + height $= f(x_i)$ (i = index).

$$A = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \dots + f(x_n)\Delta x_n$$

If we divide our $[a,b]$ interval into n equal subintervals, then $\Delta x_1 = \Delta x_2 = \dots = \Delta x_n = \frac{b-a}{n}$ and

$x_1 = a + \frac{b-a}{n}$ $\quad x_2 = x_1 + \frac{b-a}{n} = a + 2\left(\frac{b-a}{n}\right)$

$\dots \quad x_i = a + i\left(\frac{b-a}{n}\right) \quad \dots \quad x_n = a + n\left(\frac{b-a}{n}\right) = b$

Riemann sum

$$\boxed{A \cong \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) \quad \text{where } x_i = a + i\left(\frac{b-a}{n}\right)}$$

If we let n continue to grow, our estimation will be better, until it becomes exact, i.e.

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}$$

☆ Area is a signed area, i.e. area above x-axis is +ve + area below x-axis is −ve.

4.2 (continued)

Defn Definite Integral

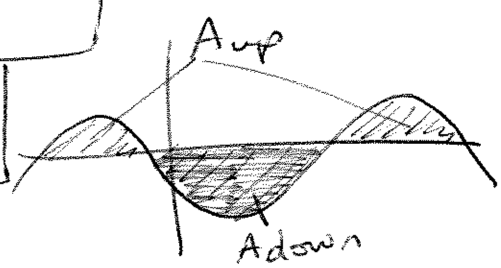
Let f be a functn that's defined on $[a,b]$. If

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i$$ exists, we say f is integrable

on $[a,b]$.

$$\underbrace{\int_a^b f(x)\,dx}_{\text{called definite integral}} = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i$$

$$\int_a^b f(x)\,dx = A_{up} - A_{down}$$



$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad a>b$$

Integrability Thm

If f is bounded on $[a,b]$ & continuous there except for a finite # of discontinuities, then f is integrable on $[a,b]$. So, if f is continuous on $[a,b]$, it's integrable on $[a,b]$.

Interval Additive Property If $f(x)$ integrable,

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

4.2 (continued)

Ex 1 Use $f(x) = 3x-1$, on the interval $[1, 3]$.
Subdivide it into 4 equal subintervals & Calculate the area of the inscribed rectangles.

Ex 2 Evaluate the definite integral using the defn. $\int_{-1}^{2} (x^2+1)\,dx$

$a=-1,\ b=2 \Rightarrow \Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$

$x_i = -1 + i\left(\frac{3}{n}\right) = -1 + \frac{3}{n}i$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i^2+1)\left(\frac{3}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left[-1+\frac{3}{n}i\right]^2+1\right)\frac{3}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 - \frac{3}{n}i - \frac{3}{n}i + \frac{9}{n^2}i^2 + 1\right)\frac{3}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 - \frac{6}{n}i + \frac{9}{n^2}i^2\right)\frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{6}{n} - \frac{18}{n^2}i + \frac{27}{n^3}i^2\right)$

$= \lim_{n\to\infty} \left[\frac{6}{n}\sum_{i=1}^{n} 1 - \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum_{i=1}^{n} i^2\right]$

$= \lim_{n\to\infty} \left[\frac{6}{n}(n) - \frac{18}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$

$= \lim_{n\to\infty} \left[6 - \frac{9(n+1)}{n} + \frac{9}{2n^2}(n+1)(2n+1)\right]$

$= \lim_{n\to\infty} \left[6 - 9 - \frac{9}{n} + \frac{9}{2n^2}(2n^2+3n+1)\right]$

$= \lim_{n\to\infty} \left[-3 - \frac{9}{n} + 9 + \frac{27}{2n} + \frac{9}{2n^2}\right]$

$= -3+9 = \boxed{6}$

4.2 (continued)

Ex3 Evaluate the definite integral using the definition. $\int_{-2}^{1} (3x^2+2)\,dx$

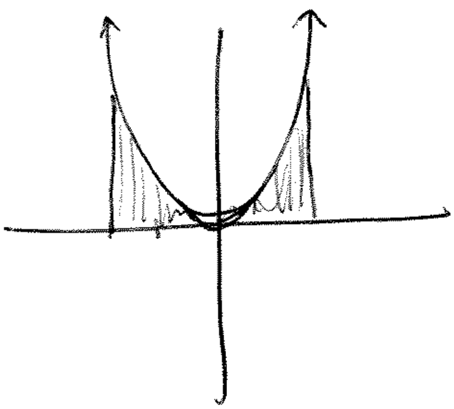
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$\Delta x =$

$x_i =$

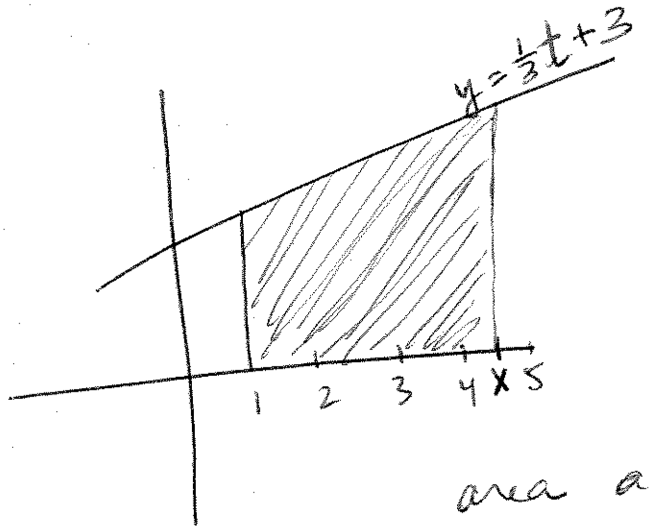
$$= \lim_{n\to\infty} \sum_{i=1}^{n} (3x_i^2+2)\Delta x$$

Ex 4 Find the area of the region under the curve of $f(x) = x^2$ on the interval $[-2, 2]$. (To do this, divide the interval $[-2, 2]$ into n equal subintervals, calulate the area of the circumscribed or inscribed rectangles + take limit as $n \to \infty$.)



4.3 The First Fundamental Theorem of Calculus

Accumulation Functions



If we want to find area of shaded region, it's

$$A=\int_1^x\left(\frac{1}{3}t+3\right)dt.$$

This is an accumulation function, because it accumulates area as we move x.

What is derivative of A?

Well, A is area of a trapezoid.

$y(1) = \frac{1}{3}+3=\frac{10}{3}$

$y(x)= \frac{1}{3}x+3$

$$\Rightarrow \quad A=\frac{1}{2}\underbrace{\left(\frac{10}{3}+\frac{1}{3}x+3\right)}_{\text{sum of bases}}\underbrace{(x-1)}_{\text{height of trapezoid}}$$

$$A=\frac{1}{2}\left(\frac{10}{3}x-\frac{10}{3}+\frac{1}{3}x^2-\frac{1}{3}x+3x-3\right)$$

$$A=\frac{1}{2}\left(\frac{1}{3}x^2+6x-\frac{19}{3}\right)$$

$$A(x)=\frac{1}{6}x^2+3x-\frac{19}{6}$$

$$\Rightarrow \quad A'(x)=\frac{1}{3}x+3$$

Aha! $\quad A'(x)=\frac{d}{dx}\left[\int_1^x\left(\frac{1}{3}t+3\right)dt\right]=\frac{1}{3}x+3$ ☺

4.3 (continued)

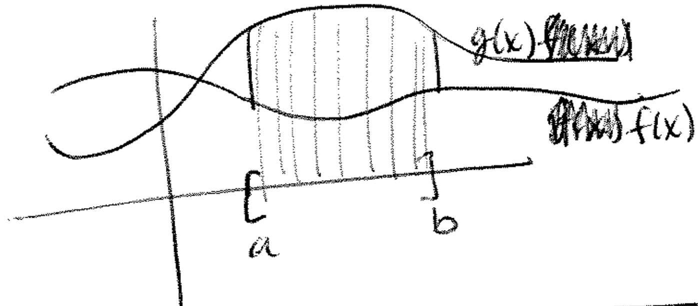
Thm First Fundamental Thm of Calculus

Let f be continuous on $[a,b]$ + let x be a value in (a,b). Then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Thm B Comparison Property

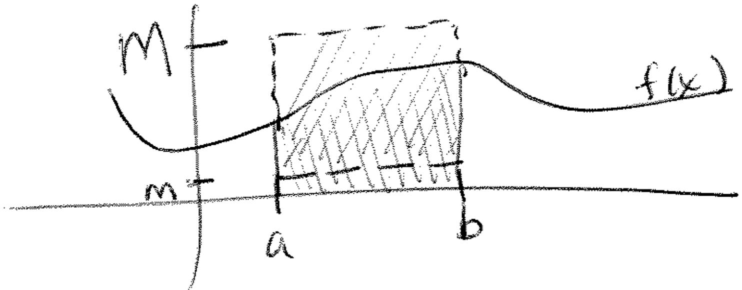
If f + g are integrable on $[a,b]$ + if $f(x) \le g(x)$ $\forall\, x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.



i.e. the area under $f(x)$ on $[a,b]$ is less than (or =) the area under $g(x)$ on $[a,b]$.

Thm Boundedness Property

If f is integrable on $[a,b]$ + $m \le f(x) \le M$ $\forall x \in [a,b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$



i.e. the area under $f(x)$ is smaller than the big rectangle but bigger than the small rectangle.

4.3 (continued)

Thm Linearity of Definite Integral

If f & g are integrable on $[a,b]$ & $k \in \mathbb{R}$,

(i) $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$

and (ii) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Ex 1 (#12) Suppose $\int_0^1 f(x)\,dx = 2$ $\int_1^2 f(x)\,dx = 3$

$\int_0^1 g(x)\,dx = -1$ and $\int_0^2 g(x)\,dx = 4$.

Calculate $\int_0^2 [\sqrt{3} f(t) + \sqrt{2} g(t) + \pi]\,dt$.

4.3 (continued)

Ex 2 Find $G'(x)$.

(a) $G(x) = \int_3^x 4t\,dt$

(b) $G(x) = \int_1^x \cos^3(2t)\tan(t)\,dt \qquad -\pi/2 < x < \pi/2$

(c) $G(x) = \int_1^x xt\,dt \qquad$ (Tricky)

4.3 (continued)

Ex 3 Find $\frac{d}{dx}\int_1^{x^2+x}\sqrt{2w+\sin w}\,dw$

4.4 The Second Fundamental Thm of Calculus + The Method of Substitution

Second Fundamental Thm of Calculus

Let f be continuous on $[a,b]$ + F be any antiderivative of f on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Ex 1 $\int_{-1}^{2} x^4\,dx$

Ex 2 $\int_{\pi/6}^{\pi/2} 2\sin t\,dt$

4.4 (continued)

Substitution Rule for Indefinite Integrals

Let g be differentiable and F = antiderivative of f. Then, if $u = g(x)$,

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u) + C$$
$$= F(g(x)) + C$$

Our friendly u-substitution is now formalized for more than the "power rule" functions.

Ex 3 $\quad \int \sqrt{x^3+1}\,(3x^2)\, dx$

Ex 4 $\quad \int_0^{\pi/2} \sin^2(3x)\cos(3x)\, dx$

4.4 (continued)

Ex 5 $\displaystyle\int_1^3 \frac{x^2+1}{\sqrt{x^3+3x}}\,dx$

Ex 6 $\displaystyle\int_{-4}^{-1} \frac{1-s^4}{2s^2}\,ds$

4.5 Mean Value Theorem (MVT) for Integrals & Symmetry

Defn Avg Value of a Function

If f is integrable on $[a,b]$, then the average value of f on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Ex 1 Find the average value of

$$f(x)=\frac{x}{\sqrt{x^2+16}} \quad \text{on } [0,3].$$

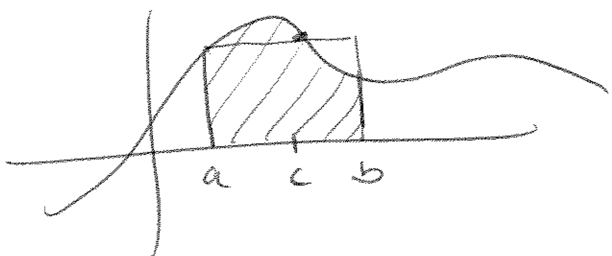
4.5 (continued)

Mean Value Thm for Integrals

If f is continuous on $[a,b]$, $\exists$ a # $c \in (a,b) \ni$
$$\int_a^b f(t)\,dt = f(c)(b-a)$$

i.e. $\exists$ some $c \in (a,b) \ni$ rectangle with height $f(c)$ + width $(b-a)$ is = area under curve on $[a,b]$

$$\Rightarrow f(c) = \frac{\int_a^b f(t)\,dt}{b-a}$$



i.e. $f(c)$ is avg (mean) value of f on $[a,b]$.

Pf let $G(x) = \int_a^x f(t)\,dt \qquad x \in [a,b]$

By MVT for derivatives (since $f(x)$ is cont, then $G(x)$ is cont. + differentiable), $\exists \; c \in (a,b) \ni$

$$\frac{G(b) - G(a)}{b-a} = G'(c)$$

$$\Leftrightarrow \quad G(b) - G(a) = G'(c)(b-a)$$

$$\int_a^b f(t)\,dt - \int_a^a f(t)\,dt = G'(c)(b-a)$$

$$\int_a^b f(t)\,dt = f(c)(b-a) \,//$$

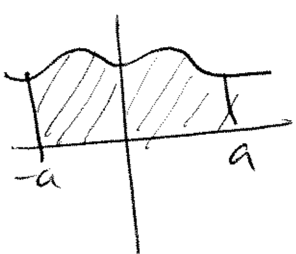
4.5 (continued)

Ex2 Find the values of c that satisfy the MVT for $f(x) = x(1-x)$ on $[0,1]$.

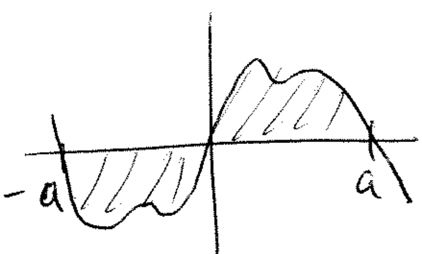
Symmetry Thm

If f is even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

If f is odd function, then $\int_{-a}^{a} f(x)\,dx = 0$.



even fn

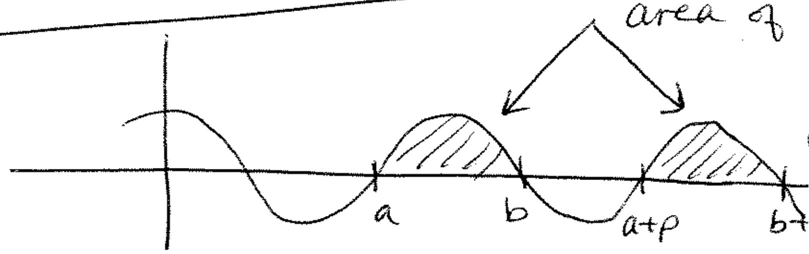


odd function

Thm P If f is periodic w/ period p, then

$$\int_{a+p}^{b+p} f(x)\,dx = \int_{a}^{b} f(x)\,dx$$

area of these are the same, they're just located differently



4.5 (continued)

Ex 3 $\int \sin(2x-4)\,dx$

Ex 4 $\int x^4 \cos(\pi x^5 - \sqrt{7})\,dx$

Ex 5 $\int x^6 (7x^7+\pi)^8 \sin\left[(7x^7+\pi)^9\right] dx$

4.5 (continued)

Ex 6 $\displaystyle\int_0^2 \frac{x^2}{(9-x^3)^{3/2}}\,dx$

Ex 7 $\displaystyle\int_0^{\pi/2} \sin x \, \sin(\cos x)\,dx$

4.5 (continued)

Ex 8 $\int_1^2 \left(1+\frac{1}{t}\right)^2 \left(\frac{1}{t^2}\right) dt$

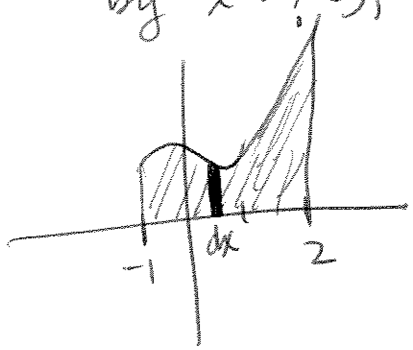
Ex 9 $\int_{-\pi/2}^{\pi/2} x^2 \sin^2(x^3) \cos(x^3)\, dx$

Ex 10 $\int_{-\pi/2}^{\pi/2} x \sin^2(x^3) \cos(x^3)\, dx$

5.1 Area of a Plane Region

$A = \left\{ \begin{array}{l} \text{area under a curve } f(x) \\ \text{from } x=a \text{ to } x=b \end{array} \right\} = \int_a^b f(x)\,dx$

Ex 1 Find area of region under $f(x) = x^3 - x + 2$ (bounded below by x-axis), on $[-1, 2]$.



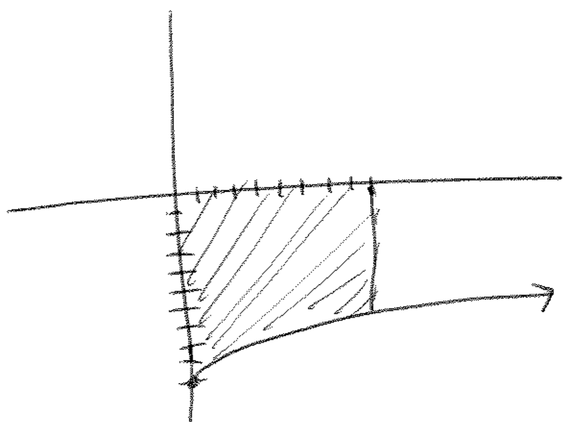
$A = \int_{-1}^{2} x^3 - x + 2 \quad dx$

Process

1. Sketch graph.
2. Slice into thin pieces + label. (decide dx or dy)
3. Decide on integration bounds.
4. Take integral of function.

5.1 (continued)

Ex 2 Find area between $y = \sqrt{x} - 10$ & $y = 0$, between $x = 0$ & $x = 9$.



5.1 (continued)

Ex 3 Find area between $y = x^2 - 9$ + $y = (2x-1)(x+3)$.

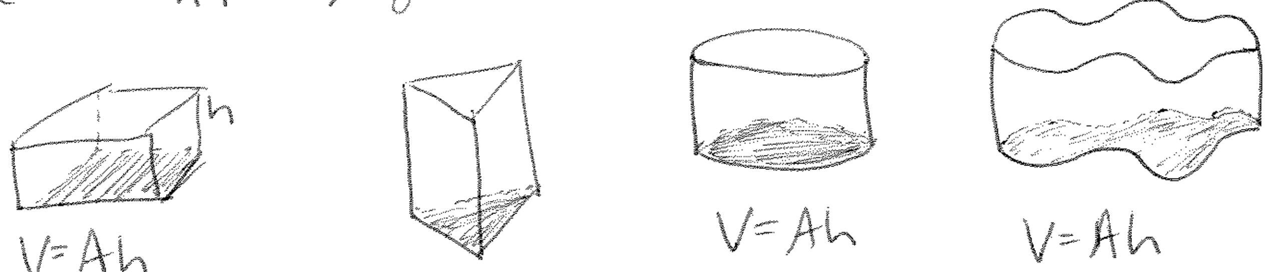
5.1 (continued)

Ex 4 Find the area of the region bounded by $x = y^2 - 2y$ and $x - y - 4 = 0$.

5.2 Volumes of Solids (Slabs, Disks, Washers)

Definite integral = ∞ sum of thin slices of something where the slice has a thickness of dx (or dy)

volume of right prisms/cylinders $\Rightarrow$



$V = Ah$ where A = area of base

$V = Ah$

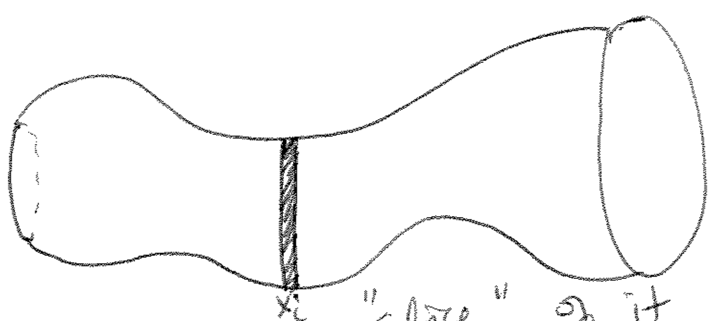
$V = Ah$

$V = Ah$

$V_{penny} = A_{penny\ face}\,\Delta x$ where Δx = thickness of penny

To find volume of a stack of n pennies,

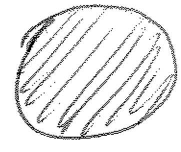
$V = \sum_{i=1}^{n} A\,\Delta x$ (looks like a Riemann sum ☺)



Imagine we have a 3d solid like this & we want to find the volume.

We can take a "slice" of it with a thickness Δx.

The very thin slice will look like



.

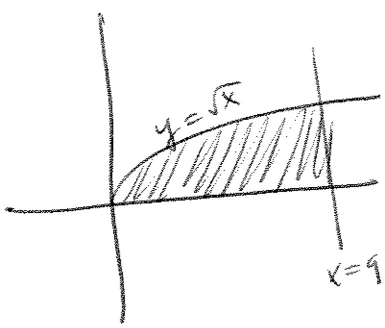
$\Rightarrow V \approx \sum_{i=1}^{n} A(x_i)\,\Delta x$ & $V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\,\Delta x$

$\Rightarrow$ $\boxed{V = \int_a^b A(x)\,dx}$ where $A(x)$ is the area of the circular slice

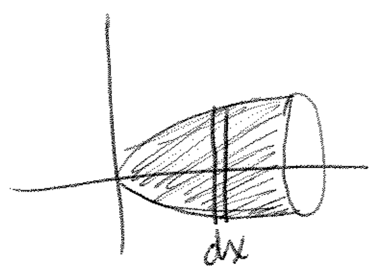
5.2 (continued)

We will now find the volume of a **solid of revolution**; i.e. a 3d solid generated by revolving a 2d curve about an axis in a 2d plane.

Ex 1 Find the volume of the solid of revolution obtained by revolving the region bounded by $y=\sqrt{x}$, the x-axis + the line $x=9$ about the x-axis.



revolved about x-axis



Each slice is  with $A=\pi r^2$.

$$\Rightarrow V=\int_0^9 \pi r^2 \, dx$$ but $r=$ vertical dist. from x-axis to $y=\sqrt{x}$

$$=\pi\int_0^9 (\sqrt{x})^2\,dx$$

$$=\pi\int_0^9 x\,dx = \frac{\pi}{2}x^2\Big|_0^9 = \frac{\pi}{2}(81-0)$$

$$=\boxed{\frac{81\pi}{2}}$$

Disk method Process

① Sketch the graph.

② Decide on a dy or dx thickness for each slice.

③ Find limits of integration.

④ Determine area functn for each slice.

⑤ Integrate! :)

Hint #④: Area $=\pi r^2$

5.2 (continued)

Ex 2 Find the volume of the solid generated by revolving the region enclosed by
$x = \frac{2}{y}$, $y = 2$, $y = 6$, $x = 0$ about the y-axis.

5.2 (continued)

<u>Ex 3</u> Find the volume of the solid generated by revolving about the x-axis the region bounded by $y = 6x$ + $y = 6x^2$.

Washer method

Each slice here is a washer rather than a circle.

So, Area $= \pi\left[r_{outer}^2 - r_{inner}^2\right]$

Otherwise, this is same as Disk method process.

5.2 (continued)

Ex 4 Find the volume of the solid generated by revolving about the line $y=2$ the region in the 1^{st} quadrant bounded by the parabolas $3x^2-16y+48=0$ + $x^2-16y+80=0$ + the y-axis. (Hint: always measure radius from axis of revolution!)

5.3 Volumes of Solids (Shells)

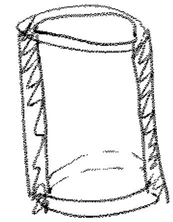
There are 2 methods used to find volume of a solid of revolution

1. Disk/Washer method
2. Shell method

[For most problems, the washer method will be easier. But, the shell method is needed for other cases where the washer method is too ugly.]

We know $V = Ah$ (A = area of base)

For a shell, then



$$\Rightarrow V = (\pi r_o^2 - \pi r_i^2)h$$

$$= \pi(r_o^2 - r_i^2)h$$

$$= \pi(r_o - r_i)(r_o + r_i)h$$

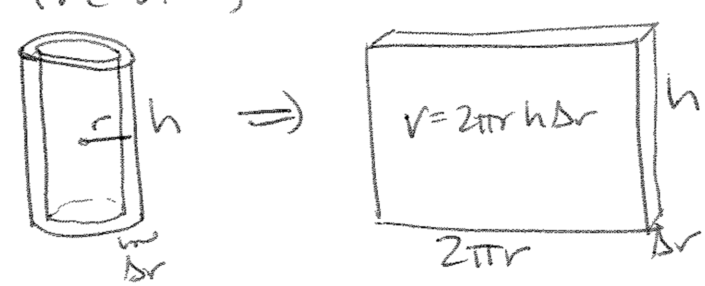
$$= 2\pi \underbrace{\left(\frac{r_o + r_i}{2}\right)}_{\text{avg radius}} \underbrace{(r_o - r_i)}_{\Delta r} h$$

$$= 2\pi r \Delta r h$$

r_o = outer radius

r_i = inner radius

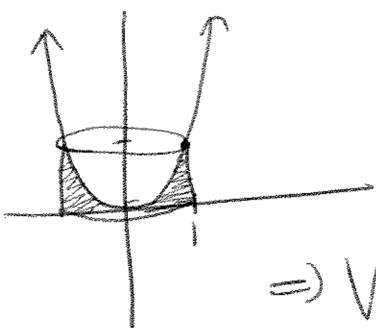
If we cut our shell down the side, we get:



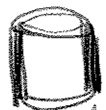
Now, we can think of adding up a bunch of "small thickness" shells to get the volume of a solid cylinder. $\boxed{V = 2\pi \int_a^b x f(x)\, dx}$

(*See nice picture on pg 288 of book)

Ex 1 Find the volume of the solid generated when the region bounded by $y=x^2$, $x=1$, $y=0$ is revolved about the y-axis. (Use the shell method.)



Our shell will be like



dx

i.e. the shell thickness is $\underline{dx}$.

$$\Rightarrow V = 2\pi \int_0^1 \text{radius} \cdot \text{ht} \quad dx$$

radius of shell = x (measured from y-axis)

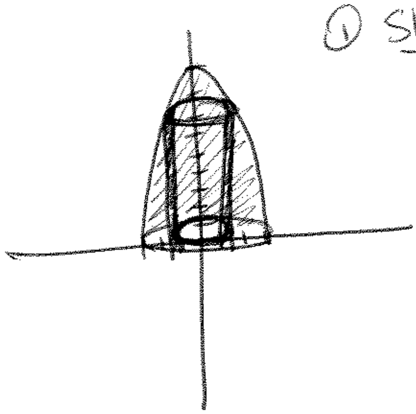
ht of shell = y (measured from x-axis)

$$\Rightarrow V = 2\pi \int_0^1 x(x^2)\,dx$$

5.3 (continued)

Ex 2 Find the volume of the solid generated when the region bounded by $y = 9 - x^2$ $(x \geq 0)$, $x = 0$, $y = 0$ is revolved about the y-axis.

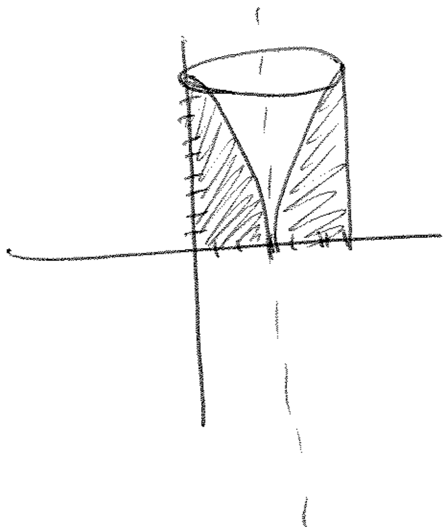
① Shell method:



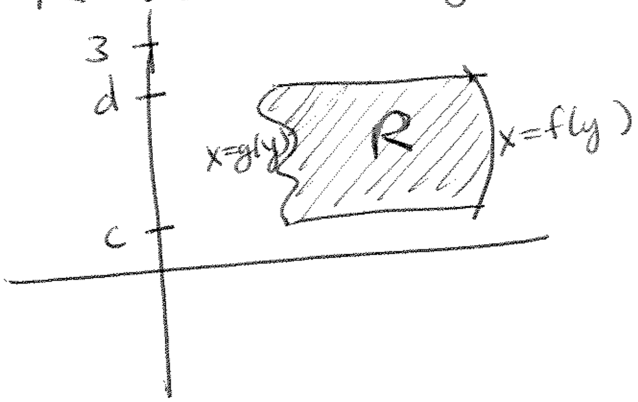
② Disk method:

5.3 (continued)

Ex 3 Find the volume of the solid generated when the region bounded by $y=9-x^2$ $(x \ge 0)$, $x=0$, $y=0$ is revolved about the line $x=3$.



Ex 4 A region R is shown below. Set up an integral for the volume obtained by revolving R about the given line.



(a) The y-axis.

(b) The x-axis.

(c) The line $y=3$.

5.4 Length of a Plane Curve

Plane curve ⇒ a curve that lies in a 2d plane.
We can define a plane curve using parametric equations; ie. by defining y + x both as functions of a parameter.

For example, we know from trig that

$y = \sin\theta$ + $x = \cos\theta$ on unit circle.

and $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow y^2 + x^2 = 1$

+ we know $x^2 + y^2 = 1$ is usual eqn for unit circle in a Cartesian coordinate system.

So, we can define a unit circle as

① $x^2 + y^2 = 1$

or ② $x = \cos\theta$ where θ is our "parameter"
$y = \sin\theta$, $\theta \in [0, 2\pi)$

Ex 1 Sketch the graph of the curve given by

$x = 3t^2 + 2$ + $y = 2t^2 - 1$ $1 \le t \le 4$

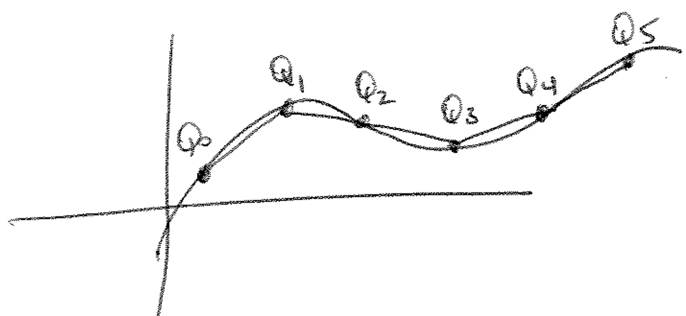
5.4 (continued)

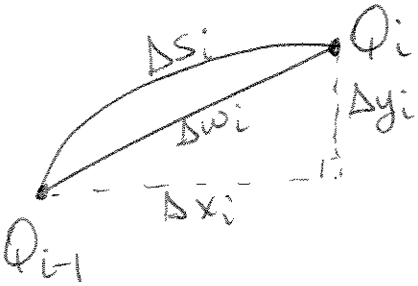
When we trace a plane curve given parametrically, we use arrowheads (visually) to indicate orientation of curve, i.e. how it travels as our parameter increases.

Defn A plane curve is smooth if it is given by a pair of parametric eqns $x = f(t)$, $y = g(t)$, $t \in [a,b]$, where f' & g' exist + are continuous on $[a,b]$, and $f'(t)$ + $g'(t)$ are not simultaneously zero on (a,b).

Arc length (typically s = arc length)

We can approximate length of a plane curve by adding up lengths of linear segments, between Q_i (pts on curve)





linear distance from Q_{i-1} to $Q_i \Rightarrow$

$$\Delta w_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

but $\Delta x_i = f(t_i) - f(t_{i-1})$
+ $\Delta y_i = g(t_i) - g(t_{i-1})$

$$\Rightarrow \Delta w_i = \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2}$$

From MVT for Derivatives, we know $\bar{t}_i + \hat{t}_i$ exist

$\ni$ $f(t_i) - f(t_{i-1}) = f'(\bar{t}_i)\Delta t_i$ w/ $\Delta t_i = t_i - t_{i-1}$

+ $g(t_i) - g(t_{i-1}) = g'(\hat{t}_i)\Delta t_i$

5.4 (continued)

$$\Rightarrow \Delta w_i = \sqrt{[f'(\bar{t}_i)\Delta t_i]^2 + [g'(\hat{t}_i)\Delta t_i]^2}$$

$$= \sqrt{([f'(\bar{t}_i)]^2 + [g'(\hat{t}_i)]^2)\Delta t_i^2}$$

$$= \sqrt{[f'(\bar{t}_i)]^2 + [g'(\hat{t}_i)]^2}\ \Delta t_i$$

Approximate length of curve $= \sum_{i=1}^{n} \Delta w_i$

$$= \sum_{i=1}^{n} \sqrt{[f'(\bar{t}_i)]^2 + [g'(\hat{t}_i)]^2}\ \Delta t_i$$

Arc length

$$\Rightarrow \boxed{L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt}$$

where L = arclength of plane curve given by $x = f(t)$ + $y = g(t)$ $a \le t \le b$

i.e. $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\ dt = \boxed{\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt}$

If $y = f(x)$ (so no parametric eqns), then

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx}.$$

Likewise, if $x = g(y)$, then

$$\boxed{L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy}$$

5.4 (continued)

Ex 1 (A classic ☺) Find the circumference of the circle $x^2+y^2=r^2$.

We can represent this w/ parametric eqns $\theta \in [0, 2\pi]$

$x = r\cos\theta \qquad y = r\sin\theta$

$\frac{dx}{d\theta} = -r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta$

$$\Rightarrow L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(\sin^2\theta + \cos^2\theta)}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2}\, d\theta$$

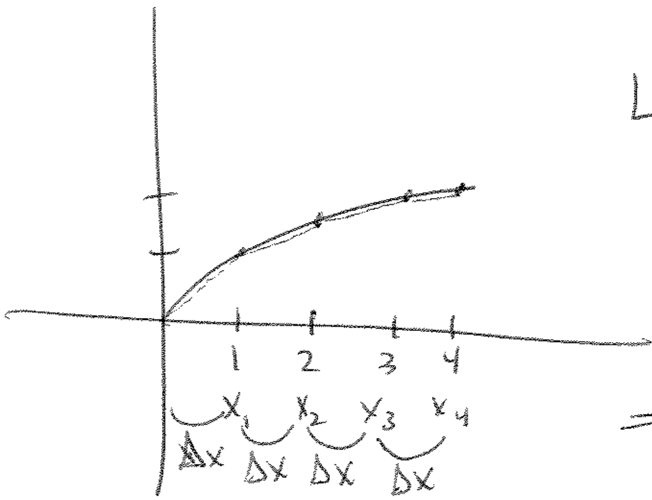
$$= \int_0^{2\pi} r\, d\theta$$

$$= r\int_0^{2\pi} d\theta$$

$$= r(\theta)\Big|_0^{2\pi} = r(2\pi - 0) = 2\pi r \quad //$$

Ex 2 Find length of line segment on $2y-2x+3=0$ between $y=1$ + $y=3$. (Check using distance formula.)

Ex 3 (a) Estimate the arc length of curve $f(x)=\sqrt{x}$ from $x=0$ to $x=4$ by 4 line segments.



$$L \simeq \sum_{i=1}^{4} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

but for this problem

$\Delta x_i = 1$ + $\Delta y_i = f(x_i) - f(x_{i-1})$

$$\Rightarrow L \simeq \sum_{i=1}^{4} \sqrt{1 + [f(x_i) - f(x_{i-1})]^2}$$

$$\Rightarrow L \simeq \sqrt{1+[\sqrt{1}-\sqrt{0}]^2} + \sqrt{1+[\sqrt{2}-\sqrt{1}]^2} + \sqrt{1+[\sqrt{3}-\sqrt{2}]^2} + \sqrt{1+[\sqrt{4}-\sqrt{3}]^2}$$

$$= \sqrt{1+1} + \sqrt{1+2-2\sqrt{2}+1} + \sqrt{1+3-2\sqrt{6}+2} + \sqrt{1+4-4\sqrt{3}+3}$$

$$= \sqrt{2} + \sqrt{4-2\sqrt{2}} + \sqrt{6-2\sqrt{6}} + \sqrt{8-4\sqrt{3}}$$

$$\simeq$$

(b) Find arc length now.

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_0^4 \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx$$

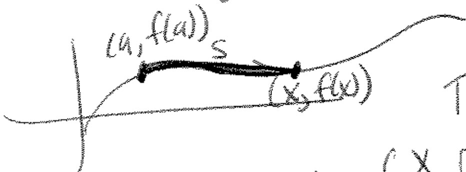
$$= \int_0^4 \sqrt{1+\frac{1}{4x}}\,dx = \int_0^4 \sqrt{\frac{4x+1}{4x}}\,dx$$

Now what?

5.4 (continued) (Surface Area)

Differential of Arc length

Let $f(x)$ be continuously differentiable on $[a,b]$. Start measuring arc length from $(a, f(a))$, up to $(x, f(x))$, where $a \in \mathbb{R}$.



Then, our arc length is a function of x.

i.e. $s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$

$$\Rightarrow s'(x) = \frac{d}{dx} \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

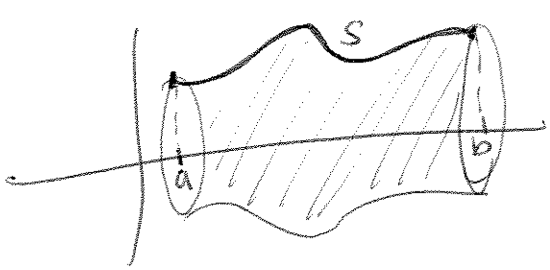
$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \boxed{ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$ arc length differential

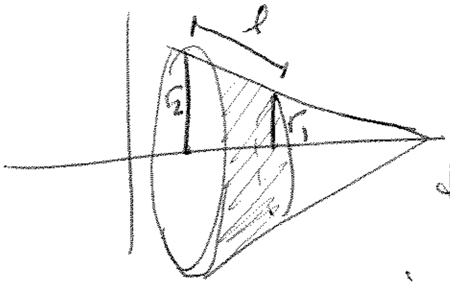
$$\boxed{\text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Surface Area (of Surface of Revolution)

Now, we'll take a plane curve + rotate it about an axis to create a hollow 3d solid. We're interested in its surface area.



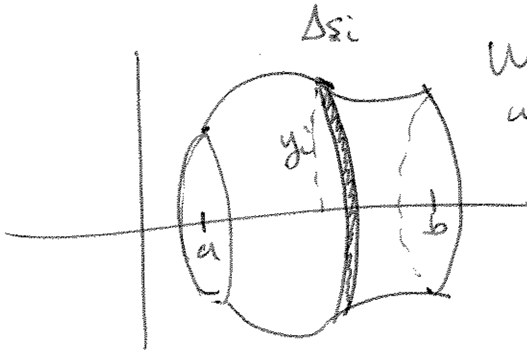
5.4 (continued)



frustum of a cone is a small piece of the cone.

& we know $A = 2\pi \left(\frac{r_1 + r_2}{2}\right)\ell$

i.e. $A = 2\pi$ (avg radius of frustum) $\cdot$ (slant ht ℓ)



We can find surface area by adding up a bunch of little frustum areas!

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi y_i \Delta s_i = \int_a^b 2\pi y \, ds$$

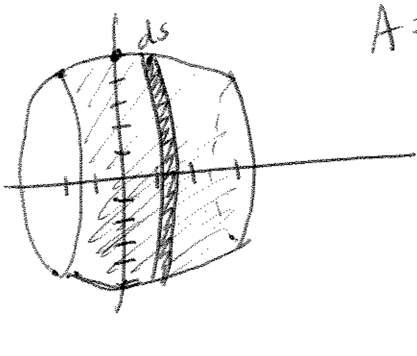
$$= \int_a^b 2\pi f(x) \, ds$$

$$\boxed{A = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx}$$

$\boxed{\text{OR} \quad A = 2\pi \int_a^b g(t) \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt \quad \text{if parametric eqns}}$

Ex 1 Find the area of the surface generated by revolving $y = \sqrt{25 - x^2}$ $\forall x \in [-2, 3]$ about the x-axis.

$$A = 2\pi \int_a^b y \sqrt{1 + (y')^2} \, dx$$



5.4 (continued)

Ex 2 Find area of the surface generated by revolving $x = 1 - t^2$, $y = 2t$, $t \in [0, 1]$ about the x-axis.

5.5 Work

Work = Force · Distance (work done by a force)

$W = FD$

(If force measured in newtons, distance in meters, then work units are joules. If force in pds + distance in ft, then work is ft-pds.)

Force is sometimes variable, in which case we need to approximate the work done in little chunks + then add up all the "chunks" of work. Aha, another case for a definite integral.

$$W = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \underbrace{F(x_i)}_{\text{Force at } x_i} \underbrace{\Delta x}_{\text{little bit of distance}} \Rightarrow \boxed{\int_a^b F(x)\,dx = W}$$

W = work
$F(x)$ = force functn

Springs

Hooke's law says $F(x) = kx$ where k = spring constant, $F(x)$ = force necessary to keep a spring stretched (or compressed) x units beyond (or short of) its natural length.

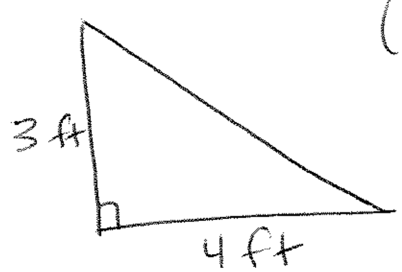
Ex 1 (#1) A force of 6 pds is required to keep a spring stretched ½ ft beyond its normal length. Find the spring constant. And find the work done in stretching the spring ½ ft beyond its natural length.

5.5 (continued)

Ex 2 A force of 1.8 newtons is required to keep a spring of natural length of 0.5 meter compressed to a length of 0.3 m. Find the work done in compressing the spring from its natural length to a length of 0.2 m.

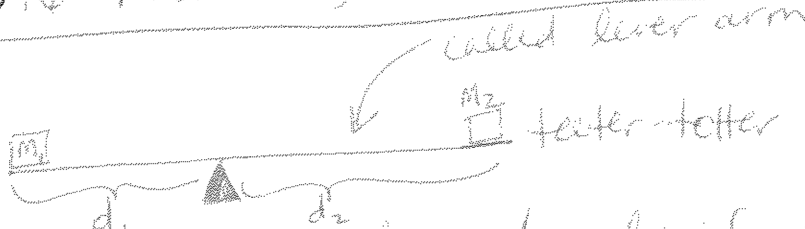
5.5 (continued)

Ex 3 (#10) A tank w/ the triangular cross section (as shown) has a length of 10 ft & is full of water. The water is to be pumped to a height of 5 feet above the top of the tank. Find the work done in emptying the tank.



(Hint: Work is still $W = Fd$ and F = weight and $\delta = 62.4\ \text{lbs}/\text{ft}^3$ is density of water.)

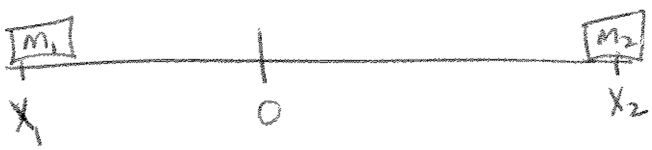
5.6 Moments, Center of Mass



m_1, m_2 masses (wts)
d_1, d_2 distance from fulcrum

This stays balanced only if $m_1 d_1 = m_2 d_2$.

If we put seesaw on x-axis w/ fulcrum at origin, then to stay balanced we need to satisfy

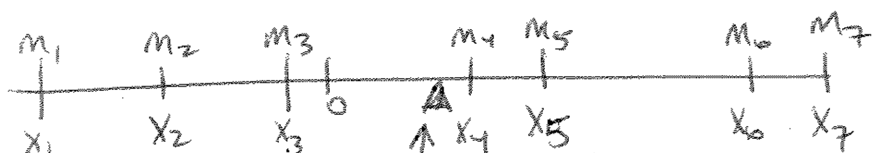


$$x_1 m_1 + x_2 m_2 = 0$$

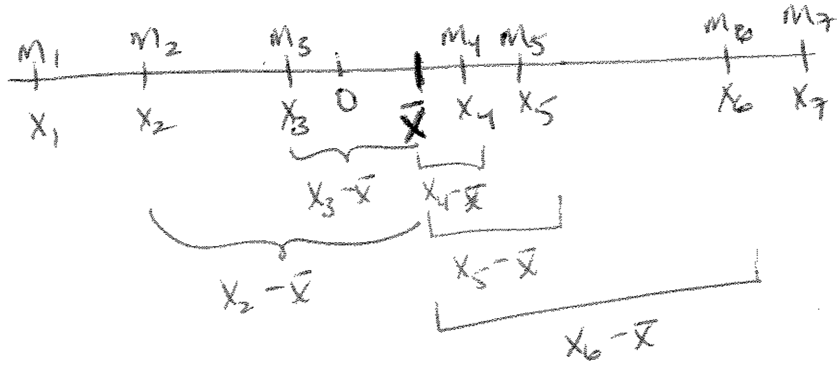
(since $x_1 = -d_1$)

<u>moment of a particle wrt a pt</u> $\Rightarrow$ product of mass m of the particle with its <u>directed</u> distance from a pt. (This measures tendency to produce a rotatn about that pt.)

total moment M for a bunch of masses $= \sum_{i=1}^{n} x_i m_i$



Where does fulcrum need to be placed to balance? Let's call it $\bar{x}$.



5.6 (continued)

Then, for balance at $\bar{x}$, we need

$$(x_1-\bar{x})m_1 + (x_2-\bar{x})m_2 + \dots + (x_n-\bar{x})m_n = 0$$

$$\iff x_1m_1 + x_2m_2 + \dots + x_nm_n = \bar{x}m_1 + \bar{x}m_2 + \dots + \bar{x}m_n$$

$$\iff x_1m_1 + x_2m_2 + \dots + x_nm_n = \bar{x}(m_1 + m_2 + \dots + m_n)$$

$$\bar{x} = \frac{x_1m_1 + x_2m_2 + \dots + x_nm_n}{m_1+m_2+\dots+m_n} = \boxed{\frac{\sum_{i=1}^{n} x_im_i}{\sum_{i=1}^{n} m_i} = \bar{x}}$$

Center of mass

i.e. balance pt.

balance pt, a.k.a. center of mass, is just M (total moment w.r.t origin) divided by m (total mass)

For a continuous mass distribution along a line (like in a wire) $\Rightarrow$

$$\boxed{\bar{x} = \frac{M}{m} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx} \quad \left(\begin{array}{l}\text{where } \delta(x)\\ = \text{density}\\ \quad \text{functn}\end{array}\right)}$$

Ex 1 John + Mary, weighing 180 lbs + 110 lbs respectively, sit at opposite ends of a 12-ft teeter totter w/ the fulcrum in the middle. Where should their 80-lb son sit in order for the board to balance?

5.6 (continued)

Ex 2 A straight wire 7 units long has density $\delta(x) = 1+x^3$ at a pt x units from one end. Find the distance from this end to the center of mass.

5.6 (continued)

Now, consider a discrete set on 2d masses.

Then, to find the center of mass (i.e. the geometric center) $(\bar{x}, \bar{y})$, we'll have

$$\bar{x} = \frac{M_y}{m} \quad + \quad \bar{y} = \frac{M_x}{m}$$

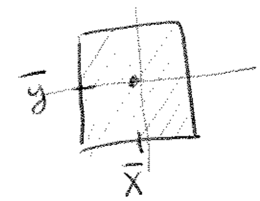
where $M_y = \sum_{i=1}^{n} x_i m_i \qquad M_x = \sum_{i=1}^{n} y_i m_i$

and $m = \sum_{i=1}^{n} m_i$

Ex 3 The masses and coordinates of a system of particles are given by the following: 5, (−3, 2); 6, (−2, −2); 2, (3, 5); 7, (4, 3); 1, (7, −1). Find the moments of this system wrt the coord. axes + find center of mass.

5.6 (continued)

Now, consider a continuous 2d region (we'll call it a lamina) that has constant (homogeneous) density everywhere. Then, to find the center of mass $(\bar{x}, \bar{y})$, we'll have (still)



$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

but $M_y = \delta \int_a^b x[f(x)-g(x)]\,dx$

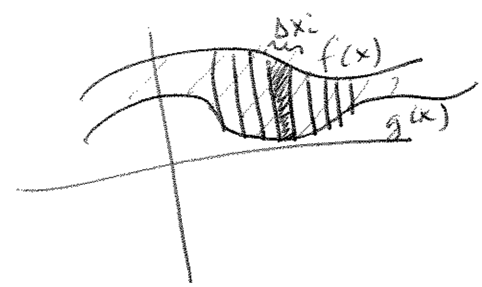
$M_x = \frac{\delta}{2} \int_a^b [f^2(x) - g^2(x)]\,dx$

and $m = \delta \int_a^b [f(x)-g(x)]\,dx$

where δ = density of lamina

because m used to be $m = \sum_{i=1}^{n} m_i$ which becomes

$m = \sum_{i=1}^{n} \delta \Delta x_i \Delta y_i = \delta \sum_{i=1}^{n} \Delta x_i (f(x_i) - g(x_i))$



$\Rightarrow M_y = \sum_{i=1}^{n} x_i m_i = \sum_{i=1}^{n} x_i (f(x_i) - g(x_i))\, \delta \Delta x_i$

$= \delta \int_a^b x\,(f(x) - g(x))\,dx$

and $\Rightarrow M_x = \sum_{i=1}^{n} y_i m_i = \sum_{i=1}^{n} \left(\overbrace{\frac{f(x_i) + g(x_i)}{2}}^{\text{avg } y\text{-value}}\right) \left(\delta\,(f(x_i) - g(x_i))\,\Delta x_i\right)$

$= \frac{\delta}{2} \sum_{i=1}^{n} [f^2(x_i) - g^2(x_i)]\,\Delta x_i$

$= \frac{\delta}{2} \int_a^b [f^2(x) - g^2(x)]\,dx$

5.6 (continued)

$$\Rightarrow \bar{x} = \frac{M_y}{m} = \frac{\delta \int_a^b x\,(f(x)-g(x))\,dx}{\delta \int_a^b [f(x)-g(x)]\,dx} = \frac{\int_a^b x\,[f(x)-g(x)]\,dx}{\int_a^b [f(x)-g(x)]\,dx}$$

and $$\bar{y} = \frac{M_x}{m} = \frac{1}{2}\left(\frac{\int_a^b [f^2(x)-g^2(x)]\,dx}{\int_a^b [f(x)-g(x)]\,dx}\right)$$

★ Note: It doesn't depend at all on density! Only depends on shape $\Rightarrow$ geometric problem

center of mass = centroid

$(\bar{x}, \bar{y})$

Ex 4 Find the centroid of the region bounded by $y = x^2$ and $y = x+2$