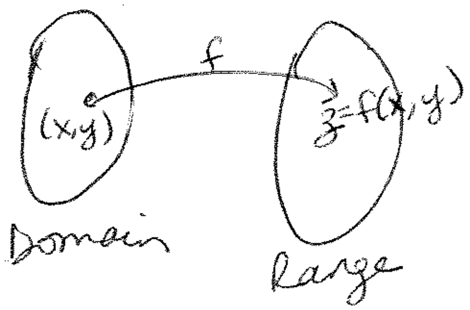
12.1 Functions of Two or More Variables

Now, we'll deal w/ functions that take 2 real inputs + give one real output.

e.g. $f(x,y) = x^2 + 3y^2$ or $g(x,y) = \sqrt{xy} + 2x^3$

These are real-valued functions of 2 $\mathbb{R}$ variables.



independent variables ⇒ x + y

dependent variable ⇒ z

domain ⇒ set of all allowable (x,y) pairs

range ⇒ set of resulting values

Ex 1 For $f(x,y) = \frac{y}{x} + xy$, find $f(1,2)$.

Find $f(a,a)$.

Find $f(\frac{1}{x}, x^2)$.

What is the domain of f?

The graph of a fn of 2 variables is a 3d surface (usually). And since its a fn, then $\exists$ only one (x, y) from the domain to each output z. Graphically, this means that each line $\perp$ to xy-plane intersects the surface in at most one pt.

Ex 2 Sketch graph of $f(x,y) = 6 - x^2$

Ex 3 sketch graph of $f(x,y) = \sqrt{16 - 4x^2 - y^2}$

12.1 (continued)

Level Curves ⇒ projectn of intersectn curves (w/ surface and planes $z=c$, $c \in \mathbb{R}$) onto xy plane.

Contour Map ⇒ a collection of level curves

(We are already used to seeing contour maps when dealing w/ temperatures across country or geographical maps.)

☆ You can use computer packages to help you visualize these graphs. Maple + Mathematica are both good + available in the computer lab.

(See examples on pg 636-637 in your book!)

Ex 4 Sketch level curves at $z = -2, -1, 0, 1, 2$

for $z = \frac{x}{y}$

12.1 (continued)

Ex 5 Draw the contour map for
$z = f(x,y) = x^2 + y$ for $z = -4, -1, 0, 1, 4$

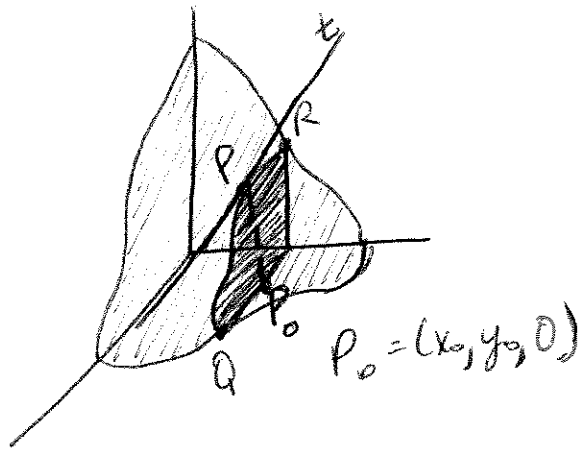
Fns of 3 or more variables

There are lots of cases where we have a fn of more than 2 variables. For instance, temperature is dependent on location which is given by 3 coords.

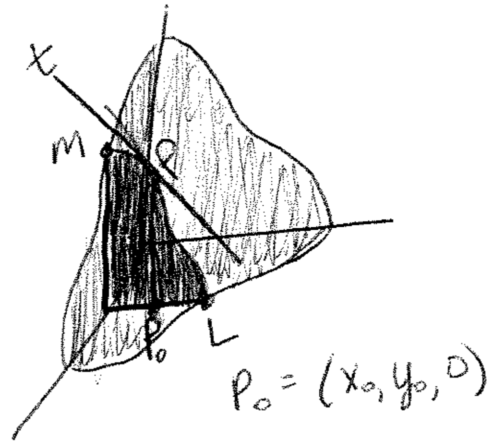
Ex 6 Find the domain for
$f(x,y,z) = \sqrt{x^2+y^2-z^2-9}$

12.2 Partial Derivatives

①



②



Consider the same surface cut by 2 different planes — in ①, it's cut by $y = y_0$ and in ②, it's cut by $x = x_0$. The curve of intersection in ① goes thru RPQ + in ②, thru MPL. Each of those curves has a tangent line associated with it at pt P. Those tangent lines have slopes associated w/ them + that should make us think about __________!

Since our function is now a fn of 2 variables (rather than 1), we can only take the <u>partial derivative</u> wrt one of the variables.

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$ slope of tangent line in ①

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$ slope of tangent line in ②

12.2 (continued)

Ex 1 Find $f_x(0,3)$ and $f_y(0,3)$ if

$$f(x,y) = 3x^2y^2 + 4y^3 - 5$$

Notation

If $z = f(x,y)$, then

$$f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$ partial derivative of f wrt x

$$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y}$$ partial derivative of f wrt y

Ex 2 If $z = x^2y + \cos(xy) - 2$, find $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$.

Ex 3 Find the slope of the tangent to the curve of intersection of the surface $3z = \sqrt{36-9x^2-4y^2}$ + the plane $x=1$ at the pt $(1, -2, \sqrt{11}/3)$.

Ex 4 The temperature in degrees Celsius on a metal plate in the xy-plane is given by $T(x,y) = 4 + 2x^2 + y^3$.
What is the rate of change of temperature wrt distance (in ft) if we start moving from $(3, 2)$ in the direction of the +ve y-axis?

12.2 (continued)

Higher Order Partial Derivatives

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \qquad f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

called "Mixed Partials"

Ex 5 Find all 4 second partial derivatives for

(a) $f(x,y) = (x^3+y^2)^5$

(b) $f(x,y) = \tan^{-1}(xy)$

Ex 6 For $f(x,y,z) = xy^2 - \frac{2x}{yz} + 3z^3x$, find

f_x, f_y, f_z, f_{xz} and f_{yy}.

12.3 Limits and Continuity

Intuitively, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ means that as the pt (x,y) gets very close to (a,b), then $f(x,y)$ gets very close to L. When we did this for fns of one variable, it could approach from only 2 sides or directions. Now, however, we can approach (a,b) from only many directions.

Defn

$\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ means that $\forall$ $\varepsilon > 0$ $\exists$ corresponding $\delta > 0$ $\ni$ $|f(x,y) - L| < \varepsilon$ provided that

$$0 < |(x,y)-(a,b)| < \delta.$$

$$|(x,y) - (a,b)| = \sqrt{(x-a)^2 + (y-b)^2}$$ i.e. the distance from (x,y) to (a,b).

So $0 < |(x,y)-(a,b)| < \delta$ means all the pts (x,y) inside a circle of radius δ, centered at (a,b). (excluding the center, since it has to be bigger than 0)

If different paths of approach lead to different limit values, then the limit does not exist.

Ex $$\lim_{(x,y)\to(3,1)} [3x^2y - x^3y^2] = 3(3^2)(1) - (3^3)(1^2)$$
$$= 27 - 27 = 0$$

12.3 (continued)

Ex 1 Find $\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2}$

Ex 2 Find $\lim_{(x,y)\to(-2,1)} (xy^3 - xy + 3y^2)$

12.3 (continued)

Ex 3 Show that $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$ does not exist.

Continuity $\Rightarrow$ a fn $f(x,y)$ is continuous at (a,b) if the ① fn exists there, ② the limit as $(x,y)\to(a,b)$ exists + if ③ $f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$.

This is basically the same as single-valued functions, of one variable.

Notice
All polynomial fns have continuity everywhere.
All rational fns are continuous everywhere the denominator is $\neq$ zero.

Composition of Fns

If a fn g of 2 variables is continuous at (a,b) + a fn f of one variable is continuous at $g(a,b)$, then $(f\circ g)(x,y) = f(g(x,y))$ is continuous at (a,b).

12.3 (continued)

Ex 4 Show that $f(x,y) = \sin(x^3 - 4xy)$ is continuous everywhere.

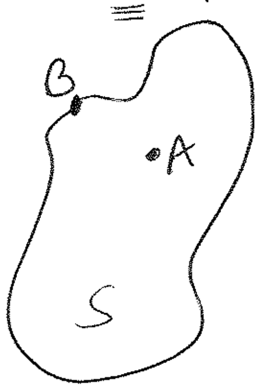
Ex 5 Determine where $f(x,y) = \ln(1-x^2-y^2)$ is continuous.

Ex 6 Is $f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & xy \neq 0 \\ 1 & xy = 0 \end{cases}$ continuous everywhere?

12.3 (continued)

A neighborhood of radius δ of pt $P \Rightarrow$ set of all pts $Q \ni |Q-P| < \delta$; i.e. set of pts inside circle centered at P w/ radius δ.
(In 3d, it means ~~an~~ the set of pts inside the sphere centered at P w/ radius δ.)

A pt R is an interior pt of a set S if there is a neighborhood of R contained in S.



A = interior pt
B = bndry pt

B is a boundary pt of S if every neighborhood of B contains pts that are in S and pts not in S.
The set of all bndry pts is called the boundary of S.

A set is open if all its pts are interior pts.
A set is closed if it contains interior pts as well as all the bndry pts.

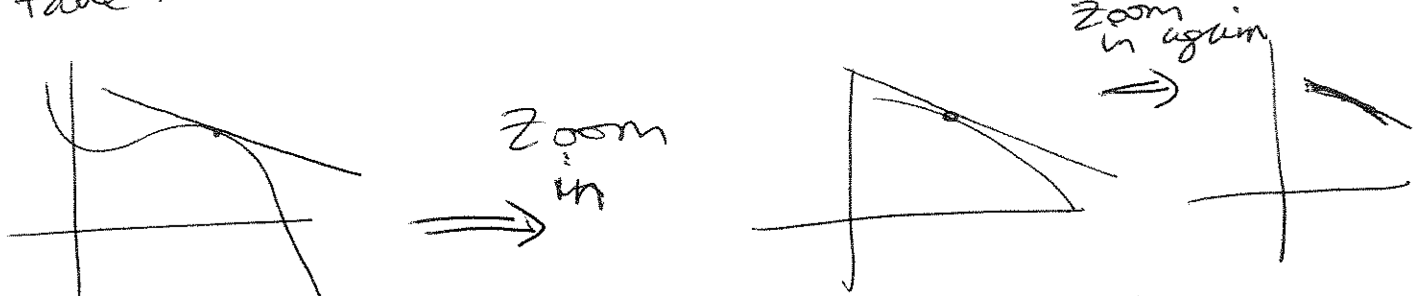
Thm Equality of Mixed Partials
If f_{xy} + f_{yx} are continuous on an open set S, then $f_{xy} = f_{yx}$ at each pt of S.

Ex 7 For $\{(x,y) : x^2+y^2 < 4\}$, sketch set. Describe boundary. Is it open, closed or neither?

12.4 Differentiability

For a fn of one variable, the derivative gave the slope of the tangent line. So, for a fn of 2 variables, we should be able to find the tangent plane using the derivative. But the first partial derivatives don't give that info.

For a fn of one variable, the tangent line approximates the fn for values very close to that pt.
$\Rightarrow$ we say f is almost linear at $x=a$, where we take the derivative.



So, if we zoom in enough, it looks like our curve is "locally linear" (ie. looks like a line) at the pt in question.

$$\Rightarrow f(a+h) = f(a) + hm + h\varepsilon(h)$$ locally linear

f is our function
a is xvalue where we're finding the slope
h is small change
m is slope of tangent line
+ $\varepsilon(h)$ is a function satisfying
$$\lim_{h\to 0} \varepsilon(h) = 0.$$

$$\Leftrightarrow \quad \varepsilon(h) = \underbrace{\frac{f(a+h)-f(a)}{h}}_{\substack{\text{slope of}\\ \text{secant line}}} - \underbrace{m}_{\substack{\text{slope of}\\ \text{tangent}\\ \text{line}}}$$

If f is locally linear at a, then

$$\lim_{h\to 0} \varepsilon(h) = \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h} - m\right] = 0$$

$$\Leftrightarrow \quad \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = m = f'(a)$$

So, if f is locally linear, then f is differentiable
+ if f is differentiable, then it's locally linear.

Now, let's see if we can leverage this equivalency for functions of 2 variables.

Defn We say f is locally linear at (a,b) if

$$\star \quad f(a+h_1, b+h_2) = f(a,b) + h_1 f_x(a,b) + h_2 f_y(a,b) + h_1 \varepsilon_1(h_1,h_2) + h_2\varepsilon_2(h_1,h_2)$$

where $\varepsilon_1(h_1,h_2) \to 0$ as $(h_1,h_2)\to 0$ and
$\varepsilon_2(h_1,h_2) \to 0$ as $(h_1,h_2)\to 0$.

let $\vec{p}_0 = (a,b)$ $\vec{h} = (h_1,h_2)$ + $\vec{\varepsilon}(\vec{h}) = (\varepsilon_1(h_1,h_2), \varepsilon_2(h_1,h_2))$

Then $\star$ becomes

$$f(\vec{p}_0 + \vec{h}) = f(\vec{p}_0) + (f_x(\vec{p}_0), f_y(\vec{p}_0))\cdot\vec{h} + \varepsilon(\vec{h})\cdot\vec{h} \quad ①$$

Defn
The function f is differentiable at $\vec{p}$ if it is locally linear at $\vec{p}$. The function f is differentiable on an open set R if it is differentiable $\forall$ pt in R.

Gradient of f

$$\nabla f(\vec{p}) = (f_x(\vec{p}), f_y(\vec{p})) = f_x(\vec{p})\hat{\imath} + f_y(\vec{p})\hat{\jmath}$$

($\nabla f(\vec{p})$ is read "del f of p".)

① becomes $\Longrightarrow$ $\boxed{f(\vec{p}+\vec{h}) = f(\vec{p}) + \nabla f(\vec{p})\cdot\vec{h} + \varepsilon(\vec{h})\cdot\vec{h}}$ ②

The gradient is the analog of a derivative for a real-valued function! (The gradient can be extended to any dimensions.)

Thm If $f(x,y)$ has continuous partial derivatives $f_x(x,y)$ & $f_y(x,y)$ on a disk D whose interior contains (a,b), then $f(x,y)$ is differentiable at (a,b).

Properties of ∇ operator

∇ is a linear operator so ⓐ $\nabla[f(\vec{p})+g(\vec{p})] = \nabla f(\vec{p}) + \nabla g(\vec{p})$

ⓑ $\nabla[\alpha f(\vec{p})] = \alpha \nabla f(\vec{p})$ $\forall \alpha \in \mathbb{R}$.

Also, ⓒ $\nabla[f(\vec{p})g(\vec{p})] = f(\vec{p})\nabla g(\vec{p}) + g(\vec{p})\nabla f(\vec{p})$.

Thm If f is differentiable at $\vec{p}$, then f is continuous at $\vec{p}$.

12.4 (continued)

Ex 1 Find the gradient ∇f.

(a) $f(x, y) = x^3y - y^3$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}$$

$$\frac{\partial f}{\partial x} = 3x^2y \qquad \frac{\partial f}{\partial y} = x^3 - 3y^2$$

$$\Rightarrow \quad \nabla f = 3x^2y\,\hat{\imath} + (x^3 - 3y^2)\hat{\jmath}$$

(b) $f(x,y) = \sin^3(x^2y)$

(c) $f(x,y,z) = xz\ln(x+y+z)$

12.4 (continued)

Tangent Plane

If f is differentiable at $\vec{p}_0$, then when $\vec{h}$ has very small magnitude

$$f(\vec{p}_0+\vec{h}) \approx f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot\vec{h}$$

Let $\vec{p} = \vec{p}_0+\vec{h} \iff \vec{h} = \vec{p}-\vec{p}_0$

Then $f(\vec{p}) \approx f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot(\vec{p}-\vec{p}_0)$

So $\boxed{T(\vec{p}) = f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot(\vec{p}-\vec{p}_0)}$

$z = T(\vec{p})$ defines a plane that approximates f near $\vec{p}_0$, i.e. the tangent plane

<u>Ex 2</u> For $f(x,y) = x^3y+3xy^2$, find eqn of tangent plane at $\vec{p}_0 = (2,-2)$

$f(2,-2) =$

$\nabla f =$

$\nabla f(2,-2) =$

$z = f(2,-2) + \nabla f(2,-2)\cdot\langle x-2, y+2\rangle =$

Ex 3 Find eqn of tangent "hyperplane" at $\vec{p}_0$.

$f(x,y,z) = xyz + x^2 \qquad \vec{p}_0 = (2,0,-3)$

Ex 4 Find all pts (x,y) at which tangent plane to graph of $z = x^3$ is horizontal

(Hint: Find tangent plane at $\vec{p}_0 = \langle x_0, y_0 \rangle$. Then, you know the normal vector & for a horizontal plane, the normal vector should be $\langle 0,0,k \rangle$ $k \in \mathbb{R}$.)

12.5 Directional Derivatives + Gradients

We know we can write

$$\frac{\partial f}{\partial x} = f_x(\vec{p}) = \lim_{h\to 0} \frac{f(\vec{p}+h\hat{\imath})-f(\vec{p})}{h}$$

$$\frac{\partial f}{\partial y} = f_y(\vec{p}) = \lim_{h\to 0} \frac{f(\vec{p}+h\hat{\jmath})-f(\vec{p})}{h}$$

The partial derivatives measure the rate of change of function <u>in the direction</u> of the x-axis or y-axis.

What about rates of change in other directions?

Defn For any unit vector $\hat{u}$, let

$$D_{\hat{u}} f(\vec{p}) = \lim_{h\to 0} \frac{f(\vec{p}+h\hat{u})-f(\vec{p})}{h}$$

If this limit exists, this is called the directional derivative of f at $\vec{p}$ in the direction of $\hat{u}$.

Thm
Let f be differentiable at $\vec{p}$. Then f has a directional derivative at $\vec{p}$ in direction of $\hat{u}$.
$\hat{u} = u_1\hat{\imath} + u_2\hat{\jmath}$ + $D_{\hat{u}} f(\vec{p}) = \vec{u} \cdot \nabla f(\vec{p})$

$$\Leftrightarrow \quad D_{\hat{u}} f(x,y) = u_1 f_x(x,y) + u_2 f_y(x,y)$$

$$= u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y}$$

12.5 (continued)

Ex 1 Find the directional derivative of at $\vec{P}_0$ in direction of $\hat{u}$ (where $\vec{a} = c\hat{u}$)

(a) $f(x,y) = y^2 \ln x \quad \vec{p} = (1,4) \quad \vec{a} = \hat{\imath} - \hat{\jmath}$

We first need to make sure we get a unit vector $\hat{u}$ (in direction of $\vec{a}$)

$$\hat{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{\langle 1,-1\rangle}{\sqrt{1+1}} = \frac{\langle 1,-1\rangle}{\sqrt{2}} = \langle 1/\sqrt{2}, -1/\sqrt{2}\rangle$$

$$\frac{\partial f}{\partial x} = \frac{y^2}{x} \quad \frac{\partial f}{\partial y} = 2y\ln x \quad \left.\frac{\partial f}{\partial x}\right|_{(1,4)} = 16 \quad \left.\frac{\partial f}{\partial y}\right|_{(1,4)} = 0$$

$$\Rightarrow D_{\hat{u}} f(x,y) = 1/\sqrt{2}\,(16) + -1/\sqrt{2}\,(0) = \frac{16}{\sqrt{2}} = 8\sqrt{2}$$

(b) $f(x,y) = 2x^2 \sin y + yx \quad \vec{p} = (1, \pi/2) \quad \vec{a} = 2\hat{\imath} + \hat{\jmath}$

12.5 (continued)

maximum rate of change

We know $D_{\vec{u}} f(\vec{p}) = \vec{u} \cdot \nabla f(\vec{p})$

$= |\vec{u}|\,|\nabla f(\vec{p})| \cos\theta$ where θ is angle between $\vec{u}$ + $\nabla f(\vec{p})$

$\Leftrightarrow D_{\vec{u}} f(\vec{p}) = |\nabla f(\vec{p})| \cos\theta$ (since $\vec{u}$ is unit vector)

$\Rightarrow D_{\vec{u}} f(\vec{p})$ is max when $\cos\theta = 1 \Leftrightarrow \theta = 0, \pi$.

i.e. when $\vec{u}$ pts in direction of gradient

Thm
A fn increases most rapidly at $\vec{p}$ in the direction of the gradient (w/ rate $|\nabla f(\vec{p})|$) and decreases most rapidly in the opposite direction (w/ rate $-|\nabla f(\vec{p})|$).

Pretty cool!

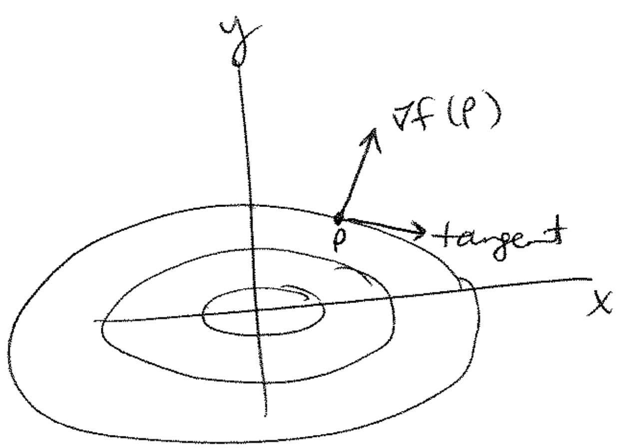
Ex 2 Find vector in direction of most rapid increase of $f(x,y) = e^y \sin x$ at $\vec{p} = (5\pi/6, 0)$. Then find the rate of change in that direction.

Ex 3 The temperature at (x,y,z) of a ball centered at the origin is $T(x,y,z) = 100e^{-(x^2+y^2+z^2)}$. Show that the direction of greatest decrease in temperature is always a vector pointing away from the origin.

12.5 (continued)

One extra (cool) fact

<u>Thm</u>
The gradient of f at pt P is $\perp$ to level curve of f through P.



Let's say we have these level curves for our function. Then, ∇f is always $\perp$ to any pt on the level curves.

12.6 The Chain Rule

<u>Thm</u> Let $x = x(t)$ & $y = y(t)$ be differentiable at t & let $z = f(x,y)$ be differentiable at $(x(t), y(t))$. Then, $z = f(x(t), y(t))$ is differentiable at t &

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \qquad \frac{dz}{dt} = \nabla f \cdot \langle x', y' \rangle$$

<u>Thm</u> Let $x = x(s,t)$ & $y = y(s,t)$ have 1st partial derivatives at $\langle s,t \rangle$ & let $z = f(x,y)$ be differentiable at $(x(s,t), y(s,t))$. Then $z = f(x(s,t), y(s,t))$ has first partial derivatives given by

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

<u>Ex 1</u> Find $\frac{dw}{dt}$ given $w = x^2 y - y^2 x$, $x = \cos t$, $y = \sin t$.
(Express answer in terms of t.)

Ex 2 Find $\frac{\partial w}{\partial t}$ given $w = \ln(x+y) - \ln(x-y)$

$x = te^{s}$ $y = e^{st}$. (Express answer in s & t.)

Ex 3 If $z = xy + x + y$ $x = r+s+t$ & $y = rst$,

find $\left.\frac{\partial z}{\partial s}\right|_{r=1, s=-1, t=2}$

12.6 (continued)

Implicit Differentiation

Let's go back to $y=f(x)$ for a moment, and assume that instead of getting y as a fn of x (explicitly), we have $F(x,y)=0$ (i.e. y is defined implicitly).

Then, we just differentiated both sides wrt x to get $\frac{dy}{dx}$.

ex $y^3-2xy+3x=4$

$$\frac{d}{dx}\left(y^3-2xy+3x\right)=\frac{d}{dx}(4)$$

$$3y^2\frac{dy}{dx}-\left(2x\frac{dy}{dx}+2y\right)+3=0$$

$$\frac{dy}{dx}\left(3y^2-2x\right)=2y-3$$

$$\frac{dy}{dx}=\frac{2y-3}{3y^2-2x}$$

Now that we have partial derivatives, we could think of this process as

$$F(x,y)=0$$

$$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0 \quad \text{+ solve for } \frac{dy}{dx}$$

Chain Rule

knowing that $\frac{dx}{dx}=1$

$$\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0 \Leftrightarrow \boxed{\frac{dy}{dx}=\frac{-\partial F/\partial x}{\partial F/\partial y}}$$

12.6 (continued)

If we expand this thinking to a fn of 2 variables, we get

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} \qquad \frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z}}$$

where $F(x,y,z)=0$ is beginning eqn.

Ex 4 If $ye^{-x} + z\sin x = 0$, find $\frac{\partial x}{\partial z}$.

12.7 Tangent Planes

We already dealt w/ tangent planes (in 15.4) to surfaces of form $z=f(x,y)$. Now, we'll do tangent planes to surfaces of form $F(x,y,z)=0$, i.e. a surface represented by any eqn in 3 variables.

If $x=x(t)$, $y=y(t)$, + $z=z(t)$, $\forall t$, then
$F(x(t), y(t), z(t))=k$ is our generic surface.

$$\Rightarrow \frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = \frac{dk}{dt} = 0$$

$$\Longleftrightarrow \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = 0$$

$$\Longleftrightarrow \nabla F \cdot \frac{d\vec{r}}{dt} = 0 \quad \text{where } \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

and remember that $\frac{d\vec{r}}{dt}$ is vector in direction of tangent line to the curve $\Rightarrow$ the gradient $\perp$ to tangent line at at ptclr pt!

Defn
Let $F(x,y,z)=k$ be a surface, F differentiable at $P(x_0,y_0,z_0)$ w/ $\nabla F(x_0,y_0,z_0) \neq \vec{0}$. Then the plane through P + $\perp$ to $\nabla F(x_0,y_0,z_0)$ is called the tangent plane to the surface at P.

Thm
For surface $F(x,y,z)=k$, eqn of tangent plane at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

$\Rightarrow F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.$

This is because P is a pt on the plane and ∇F is the normal. So, by the defn of a plane eqn, we get the above result.

Ex 1 Find the eqn of the tangent plane to
$8x^2+y^2+8z^2=16$ at $(1, 2, \sqrt{2}/2)$.

Ex 2 Find the parametric eqns of the line that is tangent to the curve of intersection of the surfaces $f(x,y,z) = 9x^2 + 4y^2 + 4z^2 - 41 = 0$

and $g(x,y,z) = 2x^2 - y^2 + 3z^2 - 10 = 0$

at the pt $(1, 2, 2)$.

12.7 (continued) (extension of 2.9 on differentials + approximations)

Defn

Let $z = f(x,y)$, f is differentiable fn, dx + dy (differentials) are variables. dz (also called total differential of f) is

$$dz = df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy = \nabla f \cdot \langle dx, dy \rangle$$

Ex 3 Use dz to approximate change in z as (x,y) moves from P to Q. Also, find Δz.

$z = x^2 - 5xy + y$ $\quad P(2,3) \quad Q(2.03, 2.98)$

12.7 (continued)

Taylor Polynomial for fns of 2 variables

$$P_2(x,y) = f(x_0,y_0) + \left[f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)\right]$$
$$+ \frac{1}{2}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0)\right.$$
$$\left. + f_{yy}(x_0,y_0)(y-y_0)^2\right]$$

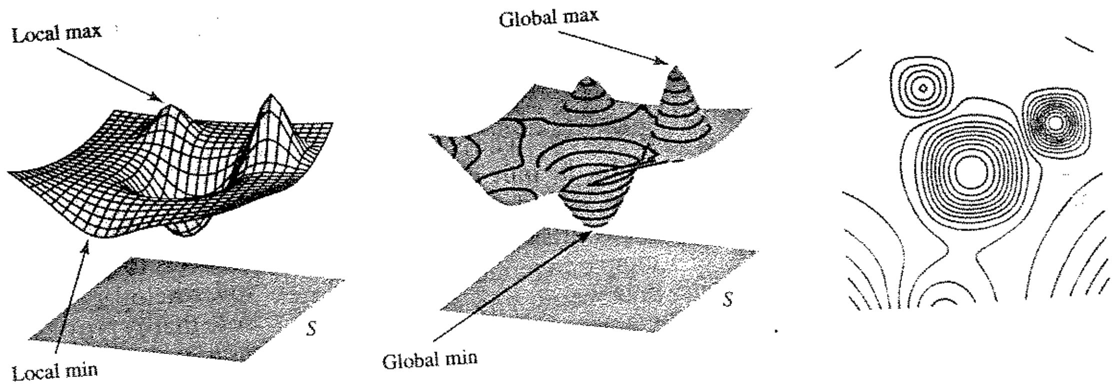
(2nd order Taylor polynomial, centered at (x_0,y_0))

Ex 4 For $f(x,y) = \tan\left(\frac{x^2+y^2}{64}\right)$ find 2nd order Taylor polynomial based at $(0,0)$. Then estimate $f(0.2,-0.3)$, using ① Taylor polynomial and ② calculator.

12.8 Maxima + Minima

Thm (Max-Min Existence)

If f is continuous on a closed, bounded set S, then f attains both a global max value + global min value there.



Thm (Critical Pt Thm)

Let f be defined on a set S containing $\vec{p}_0$. If $f(\vec{p}_0)$ is an extreme value, then $\vec{p}_0$ must be a critical pt, i.e. either $\vec{p}_0$ is

(1) a bndry pt of S

or (2) a stationary pt of S (pt $\vec{p}_0$ where $\nabla f(\vec{p}_0) = \vec{0}$) (where tangent plane is horizontal)

or (3) a singular pt of S. (a pt where f is not differentiable)

12.8 (continued)

Second Partials Test Thm

Suppose $f(x,y)$ has continuous second partial derivatives in neighborhood of (x_0, y_0) & $\nabla f(x_0,y_0)=\vec{0}$.

Let $D = D(x_0,y_0) = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0)$

Then ① if $D > 0$, $f_{xx}(x_0,y_0) < 0$, $f(x_0,y_0)$ local max

② if $D > 0$, $f_{xx}(x_0,y_0) > 0$, $f(x_0,y_0)$ local min

③ if $D < 0$, $f(x_0,y_0)$ not an extreme value
((x_0,y_0) is saddle pt.)

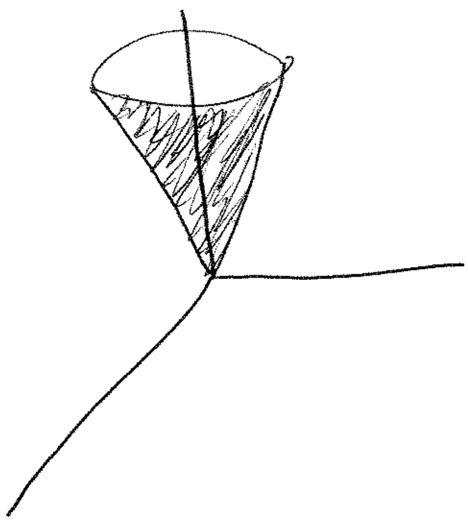
④ if $D = 0$, test is inconclusive.

Ex 1 For $f(x,y) = xy^2 - 6x^2 - 3y^2$, find all critical pts. Indicate whether each is min, max or saddle pt.

Ex 2 Find global max value + min value of

$f(x,y) = x^2 + y^2$ on $S = \{(x,y) \mid x \in [-1,3],\ y \in [-1,4]\}$

+ pts that yield those min + max values.



$z = x^2 + y^2$
is right
circular cone

12.8 (continued)

Ex 3 Find global max + min pts for

$f(x,y) = x^2 - 6x + y^2 - 8y + 7$ on $S = \{(x,y) \mid x^2 + y^2 \le 1\}$

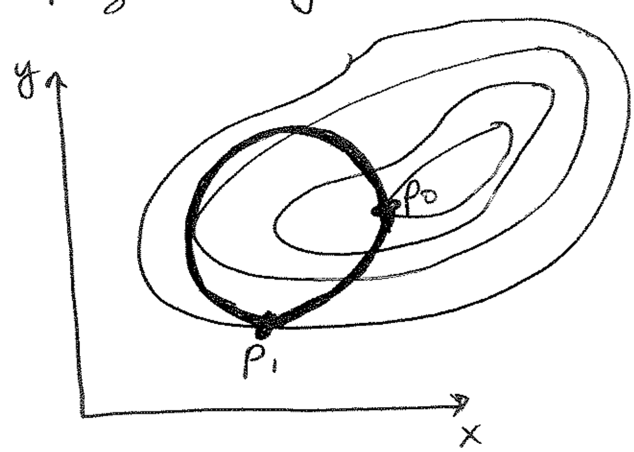
12.8 (continued)

Ex 4 Find the 3d vector w/ length 9 & whose sum of its components is a max.

12.9 Lagrange's Method

We saw (in last section) examples of constrained extremum problems, i.e. max or min problems w/ some condition/constraint. Now, we'll see easier way to solve those. :)

Optimize $f(x,y)$ subject to constraint $g(x,y)=0$.



Graphically, let ⌒ curves be level curves of $f(x,y)$, i.e. where $f(x,y)=k$, k constant. And ▬ curve is constraint curve.

To maximize f subject to $g(x,y)=0$ means to find the level curve of f w/ greatest k-value that intersects constraint curve. It will be a place where two curves are tangent! (likewise for the minimum value of f.)

$\Rightarrow$ Two curves have a common $\perp$ line (if they're tangent at that pt)

And, we know the ∇f is $\perp$ to its level curves! ∇g is also $\perp$ to constraint curve.

12.9 (continued)

$\Rightarrow \nabla f$ + ∇g are parallel at $\vec{p}_0$ + $\vec{p}_1$

$\Leftrightarrow \nabla f(\vec{p}_0) = \lambda_0 \nabla g(\vec{p}_0)$ and $\nabla f(\vec{p}_1) = \lambda_1 \nabla g(\vec{p}_1)$

$\lambda_0, \lambda_1 \in \mathbb{R}$ $\lambda_0 \neq 0$, $\lambda_1 \neq 0$.

Thm Lagrange's method

To maximize or minimize $f(\vec{p})$ subject to constraint $g(\vec{p})=0$, solve system of eqns

$$\nabla f(\vec{p}) = \lambda \nabla g(\vec{p}) \quad + \quad g(\vec{p}) = 0$$

for $\vec{p}$ + λ. Each pt $\vec{p}$ is a critical pt for constrained extremum problem + corresponding λ is called Lagrange multiplier.

Ex 1 Find max of $f(x,y) = xy$ subject to constraint $g(x,y) = 4x^2 + 9y^2 - 36 = 0$.

Ex 1 (cont.)

Ex 2 Find the least distance between the origin & the plane $x+3y-2z=4$.

12.9 (continued)

If we have more than one constraint, additional Lagrange multipliers are used.
If we want to maximize $f(x,y,z)$ subject to $g(x,y,z)=0$ and $h(x,y,z)=0$, then we solve

$\nabla f = \lambda \nabla g + \mu \nabla h \quad g=0 \ \& \ h=0.$

Ex 3 Find the max volume of the 1^{st}-octant rect. box (w/ faces $\parallel$ to coordinate planes) w/ one vertex at $(0,0,0)$ & diagonally opposite vertex on plane $3x-y+2z=1$

<u>Ex 4</u> Find minimum distance from origin to the line of intersection of the 2 planes

$x+y+z=8$ & $2x-y+3z=28$.

$f(x,y,z)=x^2+y^2+z^2$

$g(x,y,z)=x+y+z-8=0$

$h(x,y,z)=2x-y+3z-28=0$