Math2210 Midterm 3 Summer, 2008 Dylan Zwick

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Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. (10 pts each) Evaluate the integrals.

(a) $\int_{-2}^{0}\int_{1}^{x}(3x^2-5y^3)\,dy\,dx$

$$\int_{-2}^{0}\int_{1}^{x}(3x^2-5y^3)dy\,dx$$

$$= \int_{-2}^{0}\left(3x^2y-\frac{5}{4}y^4\Big|_1^x\right)dx = \int_{-2}^{0}\left[\left(3x^3-\frac{5}{4}x^4\right)-\left(3x^2-\frac{5}{4}\right)\right]dx$$

$$= \int_{-2}^{0}\left(3x^3-\frac{5}{4}x^4-3x^2+\frac{5}{4}\right)dx = \frac{3}{4}x^4-\frac{x^5}{4}-x^3+\frac{5}{4}x\Big|_{-2}^{0}$$

$$= 0-\left(12+8+8-\frac{5}{2}\right) = \frac{5}{2}-28 = -51/2$$

Answer 1(a): $\boxed{-51/2}$

(b) $\int_{0}^{\frac{\pi}{3}}\int_{0}^{1-\cos\theta} r\tan\theta\,dr\,d\theta$

$$\int_{0}^{\pi/3}\int_{0}^{1-\cos\theta} r\tan\theta\,dr\,d\theta$$

$$= \int_{0}^{\pi/3}\left(\frac{r^2}{2}\tan\theta\Big|_0^{1-\cos\theta}\right)d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/3}(1-2\cos\theta+\cos^2\theta)\tan\theta\,d\theta$$

$$= \frac{1}{2}\ln(\sec\theta)+\cos\theta+\frac{1}{4}\sin^2\theta\Big|_0^{\pi/3}$$

$$= \frac{1}{2}\ln(2)+\frac{1}{2}+\frac{3}{16}-1 = \frac{1}{2}\ln(2)-\frac{5}{16}$$

Answer 1(b): $\boxed{\frac{1}{2}\ln(2)-\frac{5}{16}}$

(Note: This is #1 continued!)

(c) $\int_0^1 \int_0^{x^2} \int_0^{x-y} 4x \, dz \, dy \, dx$

$$\int_0^1 \int_0^{x^2} \int_0^{x-y} 4x \, dz \, dy \, dx$$

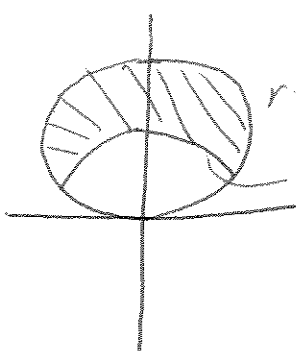
$$= \int_0^1 \int_0^{x^2} \left(4xz \Big|_0^{x-y}\right) dy \, dx = \int_0^1 \int_0^{x^2} (4x^2 - 4xy) \, dy \, dx$$

$$= \int_0^1 \left(4x^2 y - 2xy^2 \Big|_0^{x^2}\right) dx = \int_0^1 (4x^4 - 2x^5) \, dx$$

$$= \frac{4}{5}x^5 - \frac{1}{3}x^6 \Big|_0^1 = \frac{4}{5} - \frac{1}{3} = \frac{12}{15} - \frac{5}{15} = 7/15$$

Answer 1(c): $\boxed{7/15}$

2. (12 pts) Find the area of the region inside the circle $r = 8\sin\theta$ and outside the circle $r = 4$.



$$8\sin\theta = 4$$

$$\Rightarrow \sin\theta = 1/2 \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

So, the area will be

$$\int_{\pi/6}^{5\pi/6} \int_4^{8\sin\theta} r \, dr \, d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \left(\frac{(8\sin\theta)^2}{2} - \frac{4^2}{2}\right) d\theta$$

$$= \int_{\pi/6}^{5\pi/6} (32\sin^2\theta - 8) \, d\theta = 16\theta - 8\sin(2\theta) - 8\theta \Big|_{\pi/6}^{5\pi/6}$$

$$= 8\theta - 8\sin(2\theta) \Big|_{\pi/6}^{5\pi/6} = \frac{16\pi}{3} + 8\sqrt{3}$$

Answer 2: $\boxed{\frac{16\pi}{3} + 8\sqrt{3}}$

3. (12 pts) If $R=((x, y): 0 \leqslant x \leqslant 6, 0 \leqslant y \leqslant 4)$ and P is the partition of R into six equal squares by the lines x = 2, x = 4, and y = 2. Approximate $\iint_R f(x, y)\,dA$ by calculating the corresponding Riemann sum $\sum_{k=1}^{6} f(\bar{x}_k, \bar{y}_k) \;\Delta\; A_k$, assuming that $(\bar{x}_k, \bar{y}_k)$ are the centes of the six squares. Take $f(x, y)=\sqrt{x+y}$.

$f(1,1)=\sqrt{2}$ $\quad f(1,3)=2$

$f(3,1)=2$ $\quad f(3,3)=\sqrt{6}$

$f(5,1)=\sqrt{6}$ $\quad f(5,3)=\sqrt{8}$

$\Delta A_k = 4$ for each square

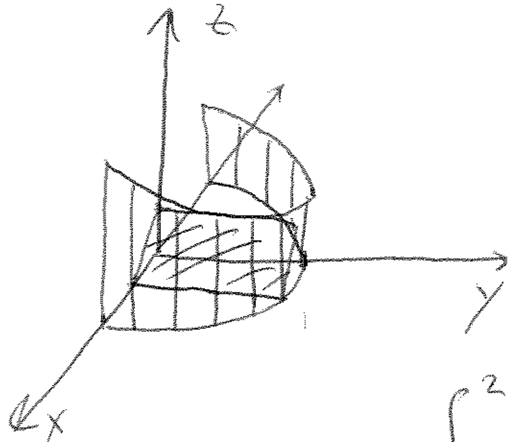
$$\sum_{k=1}^{6} f(\bar{x}_k, \bar{y}_k)\,\Delta A_k$$

$$= (\sqrt{2}+2+\sqrt{6}+2+\sqrt{6}+\sqrt{8})4$$

$$= 4\sqrt{2}+16+8\sqrt{6}+4\sqrt{8}$$

Answer 3: $\boxed{4\sqrt{2}+16+8\sqrt{6}+4\sqrt{8}}$

4. (12 pts) Find the volume of the solid in the first octant bounded by the plane $x+z=2$ and $y=6-x^2$.



$0 \le x \le 2$
$0 \le y \le 6-x^2$
$0 \le z \le 2-x$

$$\int_0^2 \int_0^{6-x^2} (2-x)\,dy\,dx$$

$$= \int_0^2 \left(2y - xy \Big|_0^{6-x^2}\right) dx$$

$$= \int_0^2 \left[2(6-x^2) - x(6-x^2)\right] dx$$

$$= \int_0^2 (12 - 6x - 2x^2 + x^3)\,dx$$

$$= 12x - 3x^2 - \frac{2}{3}x^3 + \frac{x^4}{4} \Big|_0^2$$

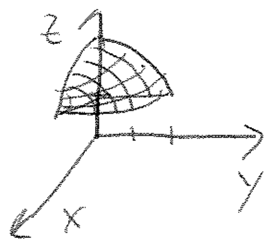
$$= 24 - 12 - \frac{16}{3} + 4$$

$$= 16 - \frac{16}{3} = \frac{32}{3}$$

Answer 4: $\boxed{32/3}$

5. For the integral $V = \int_0^2 \int_3^{7-y^2} \int_0^{\sqrt{7-y^2-z}} dx\, dz\, dy$, do the following.

(a) (8 pts) Rewrite it, changing the order of integration to $dz\, dy\, dx$. (Don't evaluate it, just set it up.)



z goes from 3 to $7-x^2-y^2$

y goes from 0 to $\sqrt{4-x^2}$

x goes from 0 to 2.

So,
$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_3^{7-x^2-y^2} dz\, dy\, dx$$

Answer 5(a): $\boxed{\int_0^2 \int_0^{\sqrt{4-x^2}} \int_3^{7-x^2-y^2} dz\,dy\,dx}$

(b) (7 pts) Rewrite it using cylindrical coordinates. (Don't evaluate it, just set it up.)

z goes from 3 to $7-x^2-y^2 = 7-r^2$

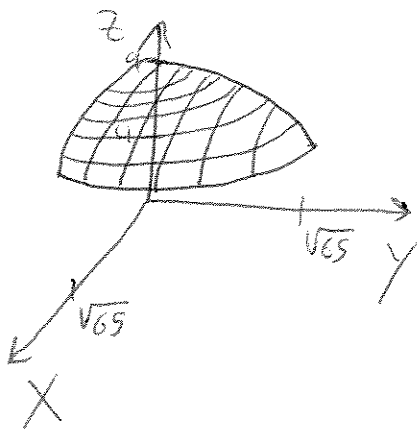
θ goes from 0 to $\pi/2$

r goes from 0 to 2

$$\Rightarrow \int_0^{\pi/2} \int_0^2 \int_3^{7-r^2} r\, dz\, dr\, d\theta$$

Answer 5(b): $\boxed{\int_0^{\pi/2} \int_0^2 \int_3^{7-r^2} r\, dz\, dr\, d\theta}$

6. (7 pts) For the integral $V = \int_0^{\sqrt{65}} \int_4^{\sqrt{81-y^2}} \int_0^{\sqrt{81-y^2-z^2}} dx\,dz\,dy$, rewrite it using spherical coordinates. (Don't evaluate it, just set it up.)



ϕ goes from 0 to

$\cos^{-1}\left(\dfrac{4}{\sqrt{4^2+\sqrt{65}^2}}\right) = \cos^{-1}\left(\frac{4}{9}\right)$

θ goes from 0 to $\pi/2$

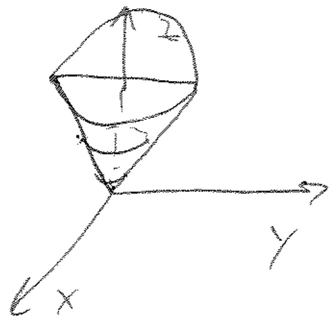
ρ goes from $\rho\cos\phi = 4$ to $\rho = 9$,
so, $4\sec\phi$ to 9.

$$\Rightarrow V = \int_0^{\cos^{-1}(\frac{4}{9})} \int_0^{\pi/2} \int_{4\sec\phi}^{9} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

Answer: $\boxed{\int_0^{\cos^{-1}(\frac{4}{9})} \int_0^{\pi/2} \int_{4/\cos\theta}^{9} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi}$

$\underset{4\sec\theta}{\parallel}$

7. (12 pts) Find the surface area of the part of the cone $z=\sqrt{x^2+y^2}$ that is cut off by (and underneath) the plane $z=4$.



$$SA = \iint_S \sqrt{f_x^2+f_y^2+1}\, dA$$

$$f_x = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$f_x^2 + f_y^2 + 1 = \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1$$

$$= 2.$$

$$\Rightarrow SA = \iint_S \sqrt{2}\, dA$$

this is most easily evaluated in polar, where $S = \begin{array}{l} 0 \le r \le 4 \\ 0 \le \theta \le 2\pi \end{array}$

$$\Rightarrow SA = \int_0^{2\pi}\int_0^4 \sqrt{2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \sqrt{2}\, \frac{r^2}{2}\Big|_0^4 d\theta = \int_0^{2\pi} 8\sqrt{2}\, d\theta$$

$$= 8\sqrt{2}\,\theta\Big|_0^{2\pi} = 16\sqrt{2}\,\pi$$

Answer: $\boxed{16\sqrt{2}\,\pi}$

Extra Credit (5 pts): Calculate $\int_{-\infty}^{\infty} e^{-x^2}\,dx$. Note that you must show your calculation, you cannot just state the answer.

$$\left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dx\,dy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta \qquad u = r^2$$

$$\frac{du}{2} = r\,dr$$

$$= \int_0^{2\pi}\int_0^{\infty} \frac{1}{2}e^{-u}\,du\,d\theta$$

$$= \int_0^{2\pi}\left(-\frac{1}{2}e^{-u}\Big|_0^{\infty}\right)d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta = \frac{\theta}{2}\Big|_0^{2\pi}$$

$$= \pi.$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

Extra Credit Answer: $\boxed{\sqrt{\pi}}$