Math2210 Midterm 2 Summer, 2008 Dylan Zwick

Name Dylan Zwick Date 07/16/08

Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. (10 pts) Find the directional derivative of $f(x,y,z) = 3xy + 2y^2z^2 - x^3$ at $p = (4,1,0)$ in the direction of $a = -2i + j + 2k$.

$$\hat{\vec{a}} = \frac{\langle -2, 1, 2 \rangle}{\sqrt{(-2)^2 + 1^2 + 2^2}} = \frac{\langle -2, 1, 2 \rangle}{3} = \langle -\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3} \rangle$$

$$\nabla f = \langle 3y - 3x^2, 3x + 4yz^2, 4y^2z \rangle$$

$$\nabla f(4,1,0) = \langle -45, 12, 0 \rangle$$

The directional derivative is:

$$\langle -45, 12, 0 \rangle \cdot \langle -\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3} \rangle$$

$$= 30 + 4 = 34$$

Answer: 34

2. (15 pts) Find a point on the surface $F(x, y, z) = 2x^2 + 3y^2 - z = 0$ where the tangent plane is parallel to the plane $8x - 3y - z = 0$.

If the tangent plane is parallel to the plane $8x-3y-z=0$ then they have parallel normal vectors. A normal vector to $8x-3y-z=0$ is $\vec{n} = \langle 8, -3, -1 \rangle$.

The normal vector to the tangent plane of the surface $F(x,y,z) = 0$ is ∇F.

$\nabla F = \langle 4x, 6y, -1 \rangle$.

So, we want

$$\langle 4y, 6y, -1 \rangle = k \langle 8, -3, -1 \rangle$$

and $2x^2 + 3y^2 - z = 0$.

This implies

$$-1 = -k$$
$$4x = 8k$$
$$6y = -3k$$

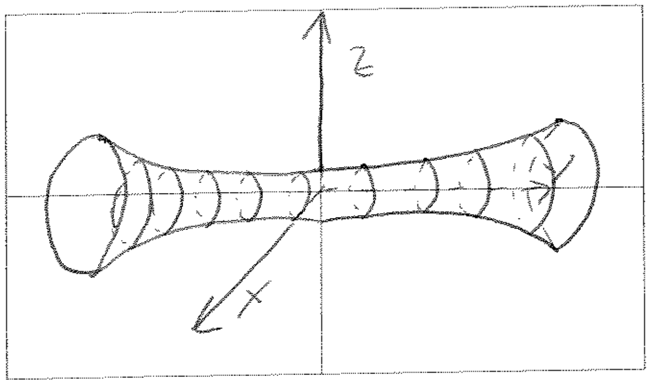
$\Rightarrow k = 1$, $x = 2$, $y = -\frac{1}{2}$, and so

$$z = 2(2^2) + 3\left(-\tfrac{1}{2}\right)^2 = 8 + \tfrac{3}{4} = \tfrac{35}{4}$$

So, the point is $\left(2, -\frac{1}{2}, \frac{35}{4}\right)$

Answer : $\left(2, -\frac{1}{2}, \frac{35}{4}\right)$

3. (10 pts) Name and sketch the graph (in 3d) for $x^2+4z^2-9=16y^2$.



Name of graph: Hyperboloid of One Sheet

Along which axis: y-axis

4. (10 pts) Describe the largest set S on which $f(x,y,z)=\ln(4-x^2-y^2-z^2)$ is continuous.

$4-x^2-y^2-z^2$ is a polynomial, so continuous everywhere. The natural logarithm is continuous when its input is positive. So, $f(x,y,z)$ is continuous when $4-x^2-y^2-z^2>0$, so, $4>x^2+y^2+z^2$. So, (x,y,z) must be inside the sphere of radius 2.

Answer: $S=\{(x,y,z): x^2+y^2+z^2<4\}$

5. (15 pts) Find all critical points for $f(x, y) = x^3 + y^3 - 2xy + 4$. Determine whether each point is a minimum, maximum or saddle point.

$$f_x(x,y) = 3x^2 - 2y$$
$$f_y(x,y) = 3y^2 - 2x$$

Note there is no boundary, and $x^3 + y^3 - 2xy + 4$ is everywhere differentiable. So, the only critical points are where $\nabla f = \vec{0}$.

$$3x^2 - 2y = 0 \Rightarrow y = \tfrac{3}{2}x^2$$
$$3y^2 - 2x = 0$$

So,

$$3\left(\tfrac{3}{2}x^2\right)^2 - 2x = 0 \Rightarrow \tfrac{27}{4}x^4 - 2x = 0$$

$\Rightarrow x\left(\tfrac{27}{4}x^3 - 2\right) = 0$. So, $x = 0$ or $x = \left(\tfrac{8}{27}\right)^{1/3} = \tfrac{2}{3}$.

So, the possible x-values are 0 and $\tfrac{2}{3}$.

The possible y-values are $y = \tfrac{3}{2}(0^2) = 0$, and $y = \tfrac{3}{2}\left(\tfrac{2}{3}\right)^2 = \tfrac{2}{3}$.

So, points are $(0,0)$ and $\left(\tfrac{2}{3}, \tfrac{2}{3}\right)$.

Now, $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -2$ $\quad D = f_{xx}f_{yy} - (f_{xy})^2$

$D = 36xy - (-2)^2 = 36xy - 4$

$D(0,0) = -4$, So, a saddle point

$D\left(\tfrac{2}{3}, \tfrac{2}{3}\right) = 36\left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right) - 4 = 16 - 4 = 12$, $f_{xx} = 6\left(\tfrac{2}{3}\right) = 4$, so a minimum.

Critical point(s) (Specify whether they're min, max or saddle.):

Minimum at $\left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{100}{27}\right)$, Saddle point at $(0,0,4)$.

6. For $z = f(x, y) = -2x^3y^2 + 5\ln(xy)$, find

(a) (10 pts) $\frac{\partial z}{\partial y}$ at (2, -1)

$$\frac{\partial z}{\partial y} = -4x^3y + 5\left(\frac{1}{xy}\right)x = -4x^3y + \frac{5}{y}$$

$$\frac{\partial z}{\partial y}(2,-1) = -4(2^3)(-1) + \frac{5}{-1}$$
$$= 32 - 5 = \boxed{27}$$

Answer : 27

(b) (10 pts) f_{xy}

$$f_x = -6x^2y^2 + 5\left(\frac{1}{xy}\right)y$$
$$= -6x^2y^2 + \frac{5}{x}$$
$$f_{xy} = \frac{\partial}{\partial y}\left(-6x^2y^2 + \frac{5}{x}\right) = \boxed{-12x^2y}$$

Answer : $-12x^2y$

7. (a) (10 pts) Convert $2x^2+2y^2=5y+81$ from a Cartesian coordinate equation into an equation in cylindrical coordinates.

$x = r\cos\theta$

$y = r\sin\theta$ $\quad 2r^2\cos^2\theta + 2r^2\sin^2\theta = 5r\sin\theta + 81$

$z = z$

$\Rightarrow 2r^2 = 5r\sin\theta + 81$

$\Rightarrow 2r^2 - 5r\sin\theta = 81$

Answer : $2r^2 - 5r\sin\theta = 81$

(b) (10 pts) Convert $\rho = -3\sec\phi$ from a spherical coordinate equation into an equation in Cartesian coordinates.

$\sec\phi = \dfrac{1}{\cos\phi}$

So, $\rho = -3\left(\dfrac{1}{\cos\phi}\right)$

$\Rightarrow \rho\cos\phi = -3$

Now, $z = \rho\cos\phi$ so

$\boxed{z = -3}$

Answer : $z = -3$

8. For $f(x, y)=x^2 \sin y+3xy-5$.

(a) (10 pts) Find ∇f .

$f_x = 2x\sin y + 3y$

$f_y = x^2\cos y + 3x$

$\nabla f = \langle 2x\sin y + 3y, x^2\cos y + 3x \rangle$

Answer : $\langle 2x\sin y + 3y, x^2\cos y + 3x \rangle$

(b) (10 pts) Find the equation of the tangent plane at (-2, 0).

$z = f(-2,0) + \nabla f(-2,0) \cdot \langle x-(-2), y \rangle$

$f(-2,0) = -5$

$f_x(-2,0) = 0$

$f_y(-2,0) = -2$

$\Rightarrow z = -5 + 0(x+2) - 2y$

$\Rightarrow \boxed{2y + z = -5}$

Answer: $2y + z = -5$

9. (10 pts) Use the total differential dz to approximate the change in z as (x, y) moves from P(-2, -0.5) to Q(-2.03, -0.51) for $z = \arctan(xy)$

$$dz = f_x\,dx + f_y\,dy \qquad dx = (-2.03 + 2) = -0.03$$
$$dy = (-0.51 - (-0.5)) = -0.01$$

$$f_x = \left(\frac{1}{1+(xy)^2}\right) y = \frac{y}{1+(xy)^2} = \frac{-\frac{1}{2}}{1+((-2)(-\frac{1}{2}))^2} = \frac{-\frac{1}{2}}{1+1} = -\frac{1}{4}$$

$$f_y = \left(\frac{1}{1+(xy)^2}\right) x = \frac{x}{1+(xy)^2} = \frac{-2}{1+((-2)(-\frac{1}{2}))^2} = \frac{-2}{2} = -1$$

$$dz = \left(-\frac{1}{4}\right)(-0.03) + (-1)(-0.01) = 0.0075 + 0.01$$
$$= \boxed{0.0175}$$

Answer: 0.0175

10. (10 pts) Find $\frac{\partial w}{\partial u}$ if $w(x, y) = x^3 - xy^2 + 2y$, $x = \sqrt{u - v}$, $y = 3uv$. (Your answer must be only in terms of u and v. Don't bother simplifying your answer.)

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial x} = 3x^2 - y^2 \qquad \frac{\partial w}{\partial y} = -2xy + 2$$

$$\frac{\partial x}{\partial u} = \frac{1}{2\sqrt{u-v}} \qquad \frac{\partial y}{\partial u} = 3v$$

$$\Rightarrow \frac{\partial w}{\partial u} = (3x^2 - y^2)\left(\frac{1}{2\sqrt{u-v}}\right) + (-2xy + 2)(3v)$$

$$= (3(u-v) - 9u^2v^2)\left(\frac{1}{2\sqrt{u-v}}\right) + \left(2 - 2\sqrt{u-v}\,(3uv)\right)3v$$

$$= (3(u-v) - 9u^2v^2)\left(\frac{1}{2\sqrt{u-v}}\right) + (6v - 18\sqrt{u-v}\,uv^2)$$

Answer: $(3(u-v) - 9u^2v^2)\left(\frac{1}{2\sqrt{u-v}}\right) + (6v - 18\sqrt{u-v}\,uv^2)$

11. (10 pts each) Find the limit, if it exists. (Show all your reasoning.)

(a) $\lim_{(x,y)\to(1,2)} \frac{8x^3 - y^3}{x^3 + 8y^3}$

It's a rational function, and the denominator $x^3 + 8y^3 \neq 0$ at $(1,2)$, so the limit is:

$$\frac{8(1^3) - 2^3}{1^3 + 8(2^3)} = \frac{0}{65} = 0$$

Answer : 0

(b) $\lim_{(x,y)\to(0,0)} \frac{x - 5y}{x + 2y}$

1) along $x = 0$

$$\lim_{y \to 0} \frac{-5y}{2y} = -\frac{5}{2}$$

2) along $y = 0$

$$\lim_{x \to 0} \frac{x}{x} = 1$$

Different, so the limit does not exist.

Answer : DNE

Extra Credit: (8 pts) Decide whether each of the following statements is either true or false.

1. $\vec{u}+(\vec{v}\times\vec{w})$ makes sense.

True or False (circle one)

2. The gradient of f is perpendicular to the graph of $z=f(x,y)$.

True or False (circle one)

3. If f is continuous at (x, y), then it's differentiable there.

True or False (circle one)

4. $(\vec{a}\cdot\vec{b})+\vec{c}$ makes sense.

True or False (circle one)

5. If $\vec{u}\cdot\vec{v}=0$ and $\vec{u}\times\vec{v}=0$, then $\vec{u}$ or $\vec{v}$ is the zero vector.

True or False (circle one)

6. If $\lim_{y\to 0} f(y,y)=L$, then $\lim_{(x,y)\to(0,0)} f(x,y)=L$.

True or False (circle one)

7. $\frac{\partial^3 f}{\partial x \partial y^2}=f_{yxy}$ always.

True or False (circle one)

8. The level curves for $5z=\sqrt{25-4x^2-y^2}$ are ellipses.

True or False (circle one)