Math2210 Midterm 1 Summer, 2008 Dylan Zwick

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Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. For $x = 2t^2 + 1$ and $y = 4t - 5$ such that $-1 \le t \le 1$, do the following:

(a) (10 pts) Eliminate the parameter to obtain the corresponding Cartesian equation.

$y + 5 = 4t \Rightarrow t = \frac{y+5}{4}$

$-9 \le y \le -1$

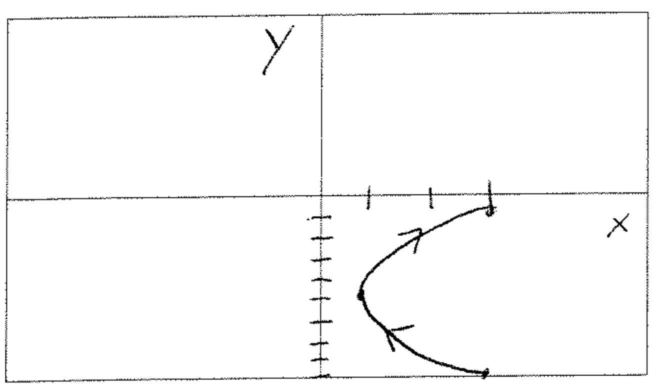
$1 \le x \le 3$

So,

$x = 2\left(\frac{y+5}{4}\right)^2 + 1 = \frac{(y+5)^2}{8} + 1$

Answer 1(a): $x = \frac{(y+5)^2}{8} + 1$

(b) (10 pts) Graph the curve.



(c) (5 pts) Indicate if the curve is simple and/or closed.

Simple: T or F (circle one)

Closed: T or F (circle one)

2. (10 pts) Find the length of the curve given by $x=\cos t$ and $y=\ln(\sec t+\tan t)-\sin t$ for $0\le t\le\frac{\pi}{4}$.

$$\frac{dx}{dt}=-\sin t\,dt \qquad \frac{dy}{dt}=\frac{\sec t\tan t+\sec^2 t}{\sec t+\tan t}-\cos t$$

$$=\sec t-\cos t$$

$$L=\int_0^{\pi/4}\sqrt{(-\sin t)^2+(\sec t-\cos t)^2}\,dt$$

$$=\int_0^{\pi/4}\sqrt{\sin^2 t+\sec^2 t-2\sec t\cos t+\cos^2 t}\;dt$$

$$=\int_0^{\pi/4}\sqrt{\sec^2 t-1}\;dt=\int_0^{\pi/4}\sqrt{\tan^2 t}\;dt$$

$$=\int_0^{\pi/4}\tan t\;dt=+\ln(\sec t)\Big|_0^{\pi/4}$$

$$=+\ln(\sqrt{2})-(+\ln(1))=\frac{\ln(2)}{2}$$

Answer 2: $\boxed{\frac{\ln(2)}{2}}$

3. (15 pts) For position vector given by $\boldsymbol{r}(t)=e^t\boldsymbol{i}+e^{-t}\boldsymbol{j}+2t\boldsymbol{k}$, find the velocity and acceleration vectors and the speed at $t=\ln 2$.

$\boldsymbol{v}(t)$ = $\vec{r}\,'(t)=e^t\hat{i}-e^{-t}\hat{j}+2\hat{k}$

$\boldsymbol{a}(t)$ = $\vec{r}\,''(t)=e^t\hat{i}+e^{-t}\hat{j}$

$$|\vec{v}(t)|=\sqrt{(e^t)^2+(-e^{-t})^2+2^2}=\sqrt{e^{2t}+e^{-2t}+4}$$

$$|\vec{v}(\ln 2)|=\sqrt{e^{2\ln 2}+e^{-2\ln 2}+4}=\sqrt{4+\frac{1}{4}+4}=\frac{\sqrt{33}}{2}$$

speed at $t=\ln 2$ = $\boxed{\sqrt{33}/2}$

4. (10 pts) Find the limit, if it exists. $\lim_{t\to 0}\left[\frac{2t\sin t}{t^2}\boldsymbol{i}-\frac{4t^3}{t^2-t}\boldsymbol{j}+\frac{\tan t}{\sin t}\boldsymbol{k}\right]$

$$\lim_{t\to 0}\frac{2t\sin t}{t^2} = \lim_{t\to 0} 2\left(\frac{\sin t}{t}\right) = 2$$

$$\lim_{t\to 0} -\frac{4t^3}{t^2-t} = \lim_{t\to 0} -\frac{4t^2}{t-1} = \lim_{t\to 0}\frac{-4(0^2)}{-1} = 0$$

$$\lim_{t\to 0}\frac{\tan t}{\sin t} = \lim_{t\to 0}\frac{\sin t/\cos t}{\sin t} = \lim_{t\to 0}\frac{1/\cos t}{1} = \frac{1/1}{1} = 1$$

So, $\lim_{t\to 0}\left[\frac{2t\sin t}{t^2}\hat{i}-\frac{4t^3}{t^2-t}\hat{j}+\frac{\tan t}{\sin t}\hat{k}\right] = \boxed{2\hat{i}+\hat{k}}$

Answer (4) : $2\hat{i} + \hat{k}$

5. (10 pts) Find the equation of the sphere that has the line segment joining (-1, 4, 3) and (3, 0, 5) as a diameter.

$$\text{Radius} = \frac{\sqrt{(3-(-1))^2+(0-4)^2+(5-3)^2}}{2} = \frac{\sqrt{4^2+4^2+2^2}}{2} = \frac{\sqrt{36}}{2} = \frac{6}{2} = 3$$

Radius = 3 units

Center at midpoint:

$$\left(\frac{3+(-1)}{2}, \frac{4+0}{2}, \frac{5+3}{2}\right) = (1, 2, 4)$$

center = (1, 2, 4)

Eqn of sphere: $\boxed{(x-1)^2+(y-2)^2+(z-4)^2 = 3^2 = 9}$

6. (10 pts each) Let $\boldsymbol{a}=\langle 2,-1,1\rangle$, $\boldsymbol{b}=\langle -3,-1,4\rangle$ and $\boldsymbol{c}=5\boldsymbol{i}+2\boldsymbol{j}$. Find each of the following.

(a) $2\boldsymbol{a}-3\boldsymbol{c}$

$$2\langle 2,-1,1\rangle-3\langle 5,2,0\rangle$$
$$=\langle 4,-2,2\rangle-\langle 15,6,0\rangle$$
$$=\langle -11,-8,2\rangle$$

$2\boldsymbol{a}-3\boldsymbol{c}$ = $\langle -11,-8,2\rangle$

(b) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$

$$\vec{b}+\vec{c}=\langle -3,-1,4\rangle+\langle 5,2,0\rangle=\langle 2,1,4\rangle$$
$$\vec{a}\cdot(\vec{b}+\vec{c})=\langle 2,-1,1\rangle\cdot\langle 2,1,4\rangle=4-1+4=7$$

$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$ = 7

(c) $\boldsymbol{b}\cdot\boldsymbol{c}-|\boldsymbol{b}|$

$$\vec{b}\cdot\vec{c}=\langle -3,-1,4\rangle\cdot\langle 5,2,0\rangle=-15-2+0=-17$$
$$|\vec{b}|=\left[\langle -3,-1,4\rangle\cdot\langle -3,-1,4\rangle\right]^{1/2}=(9+1+16)^{1/2}=\sqrt{26}$$
$$\Rightarrow \vec{b}\cdot\vec{c}-|\vec{b}|=-17-\sqrt{26}$$

$\boldsymbol{b}\cdot\boldsymbol{c}-|\boldsymbol{b}|$ = $-17-\sqrt{26}$

(Note: This is # 6 continued.) $a = \langle 2, -1, 1 \rangle$, $b = \langle -3, -1, 4 \rangle$ and $c = 5i + 2j$

(d) $\hat{c}$ (the unit vector)

$$|\vec{c}| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$$\hat{\vec{c}} = \left\langle \frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}, 0 \right\rangle$$

$\hat{c}$ = $\left\langle \frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}, 0 \right\rangle$

(e) $a \times (b \times c)$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -1 & 4 \\ 5 & 2 & 0 \end{vmatrix} = -8\hat{i} + 20\hat{j} - 1\hat{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ -8 & 20 & -1 \end{vmatrix} = -19\hat{i} - 6\hat{j} + 32\hat{k}$$

$a \times (b \times c)$ = $\langle -19, -6, 32 \rangle$

(f) $a \cdot (b \times c)$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \langle 2, -1, 1 \rangle \cdot \langle -8, 20, -1 \rangle$$
$$= -16 - 20 - 1 = -37$$

$a \cdot (b \times c)$ = -37

7. (10 pts each) For $a = i - 3j + 4k$ and $b = 2i - k$, find each of the following:

(a) Direction cosines for a.

$$\cos\alpha = \frac{a_1}{|\vec{a}|} = \frac{1}{\sqrt{1^2+(-3)^2+4^2}} = \frac{1}{\sqrt{26}}$$

$\cos\alpha$ = $\frac{1}{\sqrt{26}}$

$$\cos\beta = \frac{a_2}{|\vec{a}|} = -\frac{3}{\sqrt{26}}$$

$\cos\beta$ = $-3/\sqrt{26}$

$$\cos\gamma = \frac{a_3}{|\vec{a}|} = \frac{4}{\sqrt{26}}$$

$\cos\gamma$ = $4/\sqrt{26}$

(b) The angle θ between a and b. (Just write a simplified expression. If you don't have a calculator just write the numerical formula for the angle.)

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\langle 1,-3,4\rangle \cdot \langle 2,0,-1\rangle}{\sqrt{1^2+(-3)^2+4^2}\sqrt{2^2+0^2+(-1)^2}}$$

$$= \frac{\cancel{2-6}\ 2-4}{\sqrt{26}\sqrt{5}} = -\frac{2}{\sqrt{130}}$$

$$\theta = \cos^{-1}\left(-\frac{2}{\sqrt{130}}\right) \approx 100.1^\circ$$ so, minimal angle is $\approx 100.1^\circ$

θ = $\cos^{-1}\left(-\frac{2}{\sqrt{130}}\right) \approx 100.1^\circ$

(c) Find the projection of b onto a.

$$\text{projection of } \vec{b} \text{ onto } \vec{a} = \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$$

$$= \frac{-2}{(\sqrt{26})^2}\vec{a} = -\frac{1}{13}\langle 1,-3,4\rangle$$

$$= \left\langle -\frac{1}{13}, \frac{3}{13}, \frac{-4}{13}\right\rangle$$

Projection of b onto a = $\left\langle -\frac{1}{13}, \frac{3}{13}, -\frac{4}{13}\right\rangle$

8. (10 pts each) For the planes given by $3x + y + z = 7$ and $5x + 3z = 13$, answer the following questions.

(a) Find the line of intersection between the planes and write that line in parametric equations.

Find 2 points on both planes. So, take $y=0$ and get $\begin{matrix} 3x+z=7 \\ 5x+3z=13 \end{matrix} \Rightarrow \begin{matrix} x=2 \\ z=1 \end{matrix}$ So, $(2,0,1)$

Another mutual point would be for $z=0$. Then we get $x=\frac{13}{5}$ and $y=7-3(\frac{13}{5})=\frac{35}{5}-\frac{39}{5}=-\frac{4}{5}$. So, $(\frac{13}{5},-\frac{4}{5},0)$ is another mutual point.

The parametric equations of the line connecting these two points is:

$$x(t)=2+t\left(\tfrac{13}{5}-2\right)=2+\tfrac{3}{5}t$$
$$y(t)=0+t\left(-\tfrac{4}{5}\right)=-\tfrac{4}{5}t$$
$$z(t)=1+t(0-1)=1-t.$$

So,

Line: $x=2+\frac{3}{5}t \quad y=-\frac{4}{5}t \quad z=1-t$

Note: There are many other possible correct parametric equations

(b) Find the equation of the plane that is perpendicular to the line of intersection and goes through the point (2, 1, 3).

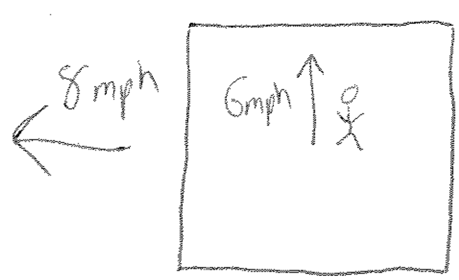
A normal vector to the plane is $\langle \frac{3}{5}, -\frac{4}{5}, -1\rangle$, which means $\langle 3,-4,-5\rangle$ is also a normal vector. Using the latter and the point $(2,1,3)$ we get:

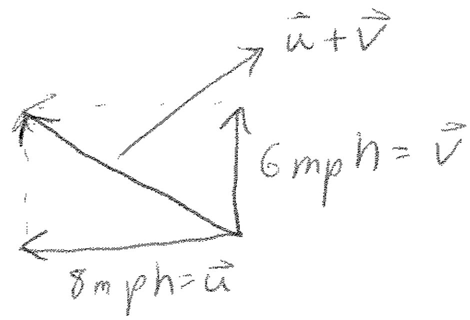
$$3(x-2)-4(y-1)-5(z-3)=0$$
$$\Rightarrow 3x-6-4y+4-5z+15=0$$
$$\Rightarrow 3x-4y-5z=-13$$
$$\Rightarrow \boxed{-3x+4y+5z=13}$$

Equation of plane: $\boxed{-3x+4y+5z=13}$

Extra Credit: (9 pts)

A luxury cruiseliner is traveling due west at only 8 miles per hour. A woman on the ship is running across the ship, heading due north, at 6 miles per hour. What are the magnitude and direction of her velocity relative to the surface of the water? (If you don't have a calculator, just give the angle in simplified form.)

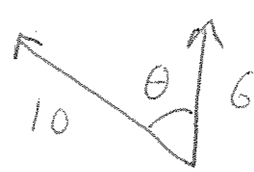




$$|\vec{u}+\vec{v}| = \sqrt{(8mph)^2+(6mph)^2} = \sqrt{100(mph)^2}$$

$$= 10 mph$$

Direction =



$$\Rightarrow \theta = \cos^{-1}\left(\frac{6}{10}\right)$$

$$= \cos^{-1}\left(\frac{3}{5}\right)$$

velocity magnitude: 10 mph

velocity direction: $\cos^{-1}\left(\frac{3}{5}\right)$ W of N.