Math2210 Final Exam Summer, 2008 Dylan Zwick

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Instructions: Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown. All answers should be completely simplified, unless otherwise stated.

1. *(15 points)* For position vector given by $\boldsymbol{r}(t)=(3t^3-5t)\boldsymbol{i}+(2t^2+7)\boldsymbol{j}$, find the velocity and acceleration vectors and the speed at $t=\frac{1}{2}$.

$$\vec{r}\,'(t) = (9t^2-5)\hat{i} + 4t\hat{j} = \vec{v}(t)$$

$\boldsymbol{v}(t) =$ $(9t^2-5)\hat{i} + (4t)\hat{j}$

$$\vec{r}\,''(t) = 18t\hat{i} + 4\hat{j} = \vec{a}(t)$$

$\boldsymbol{a}(t) =$ $18t\hat{i} + 4\hat{j}$

$$\vec{v}\left(\tfrac{1}{2}\right) = \left(\tfrac{9}{4}-5\right)\hat{i} + \left(4\left(\tfrac{1}{2}\right)\right)\hat{j}$$

$$= -\tfrac{11}{4}\hat{i} + 2\hat{j}$$

$$\left\|\vec{v}\left(\tfrac{1}{2}\right)\right\| = \sqrt{\left(-\tfrac{11}{4}\right)^2 + 2^2} = \sqrt{\tfrac{121}{16}+4} = \sqrt{\tfrac{121}{16}+\tfrac{64}{16}}$$

$$= \sqrt{\tfrac{185}{16}}$$

speed at $t=\frac{1}{2}$ = $\frac{\sqrt{185}}{4}$

2. *(20 points)* Let $\boldsymbol{a}=\langle 1,-3\rangle$, $\boldsymbol{b}=\langle 2,6\rangle$ and $\boldsymbol{c}=\langle -2,5\rangle$. Find each of the following.

(a) $2\boldsymbol{a}-3\boldsymbol{c}$

$2\vec{a} = \langle 2,-6\rangle$

$3\vec{c} = \langle -6,15\rangle$

$2\vec{a}-3\vec{c} = \langle 2,-6\rangle - \langle -6,15\rangle$

$= \langle 8,-21\rangle$

$2\boldsymbol{a}-3\boldsymbol{c}$ = $\langle 8,-21\rangle$

(b) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$

$\vec{b}+\vec{c} = \langle 0,11\rangle$

$\vec{a}\cdot(\vec{b}+\vec{c}) = \langle 1,-3\rangle\cdot\langle 0,11\rangle = -33$

$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$ = -33

(c) projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$

projection of $\vec{a}$ onto $\vec{b}$

$= \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b}$

$\vec{a}\cdot\vec{b} = \langle 1,-3\rangle\cdot\langle 2,6\rangle = 2-18 = -16$

$\|\vec{b}\|^2 = 2^2+6^2 = 40$

$= -\dfrac{16}{40}\langle 2,6\rangle = -\dfrac{2}{5}\langle 2,6\rangle$

projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ = $-\dfrac{2}{5}\langle 2,6\rangle$

(d) $\hat{\boldsymbol{a}}$ (the unit vector)

$|\vec{a}| = \sqrt{1^2+(-3)^2} = \sqrt{10}$

$\hat{\boldsymbol{a}}$ = $\dfrac{1}{\sqrt{10}}\langle 1,-3\rangle$

3. For the points A(-1, 2, 3), B(4, 1, 5) and C(1, 1, -1)

(a) *(10 points)* Write the equation of the plane through points A, B and C.

$$\vec{AB} = \langle 4-(-1),\ 1-2,\ 5-3 \rangle = \langle 5, -1, 2 \rangle$$

$$\vec{AC} = \langle 1-(-1),\ 1-2,\ -1-3 \rangle = \langle 2, -1, -4 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -1 & 2 \\ 2 & -1 & -4 \end{vmatrix} = 6\hat{i} + 24\hat{j} - 3\hat{k}$$

a parallel vector is
$2\hat{i} + 8\hat{j} - \hat{k}$

$$2x + 8y - z = D$$

$$2(-1) + 8(2) - 3$$
$$= -2 + 16 - 3$$
$$= 11$$

plane normal vector = $2\hat{i} + 8\hat{j} - \hat{k}$

Eqn of plane: $2x + 8y - z = 11$

(b) *(10 points)* Write a set of parametric equations for the line through point B and perpendicular to the plane in part (a).

$$x(t) = 2t + 4$$
$$y(t) = 8t + 1$$
$$z(t) = -t + 5$$

Line: $x = 2t + 4 \quad y = 8t + 1 \quad z = -t + 5$

4. *(15 points)* Find the directional derivative of $f(x,y,z)=x^3y-y^2z^2$ at $\boldsymbol{p}=(-1,3,-2)$ in the direction of $\boldsymbol{a}=\boldsymbol{i}-2\boldsymbol{k}$.

$$\nabla f = \langle 3x^2y,\ x^3-2yz^2,\ -2y^2z \rangle$$

$$\nabla f(-1,3,-2) = \langle 9,\ -1-2(3)(-2)^2,\ -2(3^2)(-2) \rangle$$

$$= \langle 9, -25, 36 \rangle$$

$$\nabla f \cdot \langle 1, 0, -2 \rangle = 9 - 72 = \boxed{-63}$$

$$\|\vec{a}\| = \sqrt{1^2+0^2+(-2)^2} = \sqrt{5}$$

$$\Rightarrow \quad D_{\hat{a}}(f) = -\frac{63}{\sqrt{5}}$$

Answer 4: $-63/\sqrt{5}$

5. *(20 points)* For the surface $F(x, y, z) = 4x^2 - 3xy + 2y^2 + 4z - z^2 = 4$

(a) Find the equation of the tangent plane at the point (1, 1, 1).

$\nabla F = \langle 8x - 3y,\ -3x + 4y,\ 4 - 2z \rangle$

$\nabla F(1,1,1) = \langle 5, 1, 2 \rangle$

$\cancel{F(1,1,1) = 4 - 3 + 2 + 4 -}$

$5(1) + 1(1) + 2(1) = 8$

$\Rightarrow 5x + y + 2z = 8$

Answer 5(a): $5x + y + 2z = 8$

(b) Find a point on the surface where the tangent plane is parallel to the plane $14x - 11y + 4z = 19$.

$8x - 3y = 14 \quad \Rightarrow \quad 32x - 12y = 56$

$-3x + 4y = -11 \quad\quad -9x + 12y = -33$

$4 - 2z = 4 \quad \Rightarrow 23x = 23 \Rightarrow x = 1$

$\Rightarrow y = -2$

$z = 0$

$(1, -2, 0)$

Answer 5(b): $(1, -2, 0)$

6. *(15 points)* Find all critical points of the function $f(x, y)=e^{x^2+y^2-4y}$.

$$\nabla f(x,y) = \langle 2xe^{x^2+y^2-4y}, (2y-4)e^{x^2+y^2-4y} \rangle$$

$e^{x^2+y^2-4y} \neq 0$ for any (x,y)

So,

$$\nabla f(x,y) = \langle 0, 0 \rangle$$

only at $(0,2)$

The function is differentiable everywhere and there is no boundary, so $(0,2)$ is the only critical point.

Answer 6: $(0,2)$

7. *(15 points)* For the solid inside $x^2+y^2=9$ bounded above by the sphere $x^2+y^2+z^2=16$ and below by the xy-plane, do the following.

(a) Set up the volume integral in Cartesian coordinates.

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{0}^{\sqrt{16-x^2-y^2}} dz\,dx\,dy$$

Answer 7(a): $\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{0}^{\sqrt{16-x^2-y^2}} dz\,dx\,dy$

(b) Rewrite the integral using cylindrical coordinates.

$$\int_{0}^{2\pi}\int_{0}^{3}\int_{0}^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta$$

Answer 7(b): $\int_{0}^{2\pi}\int_{0}^{3}\int_{0}^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta$

(c) Evaluate the volume (using the integral of your choice).

$$\int_{0}^{2\pi}\int_{0}^{3} r\sqrt{16-r^2}\,dr\,d\theta \qquad u=16-r^2 \qquad -\frac{du}{2}=r\,dr$$

$$=\int_{0}^{2\pi}\int_{16}^{7} -\frac{1}{2}\sqrt{u}\,du\,d\theta$$

$$=\int_{0}^{2\pi}\frac{u^{3/2}}{3}\Big|_{7}^{16}d\theta=\int_{0}^{2\pi}\left(\frac{64}{3}-\frac{7^{3/2}}{3}\right)d\theta=\frac{2\pi}{3}\left(64-7^{3/2}\right)$$

Answer 7(c): $\frac{2\pi}{3}\left(64-7^{3/2}\right)$

8. *(20 points)* Given $\boldsymbol{F}(x, y, z) = 5x^2 y z^2 \boldsymbol{i} - 2y x^3 \boldsymbol{j} + y^4 z^3 \boldsymbol{k}$, calculate the following.

(a) div $\boldsymbol{F}$

$$\text{div } \vec{F} = 10xyz^2 - 2x^3 + 3y^4z^2$$

div $\boldsymbol{F}$ = $10xyz^2 - 2x^3 + 3y^4z^2$

(b) curl $\boldsymbol{F}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5x^2yz^2 & -2yx^3 & y^4z^3 \end{vmatrix} = 4y^3z^3\,\hat{i} + (10x^2yz)\,\hat{j} + (-6yx^2 - 5x^2z^2)\hat{k}$$

curl $\boldsymbol{F}$ = $4y^3z^3\hat{i} + 10x^2yz\,\hat{j} - (6yx^2 + 5x^2z^2)\hat{k}$

(c) $\nabla(\nabla \cdot \boldsymbol{F})$

$$\nabla(\nabla \cdot \vec{F}) = (10yz^2 - 6x^2)\,\hat{i} + (10xz^2 + 12y^3z^2)\,\hat{j} + (20xyz + 6y^4z)\,\hat{k}$$

$\nabla(\nabla \cdot \boldsymbol{F})$ = $\langle 10yz^2 - 6x^2, 10xz^2 + 12y^3z^2, 20xyz + 6y^4z \rangle$

(d) $\nabla \cdot (\nabla \times \boldsymbol{F})$

$$\nabla \cdot (\nabla \times \vec{F}) = 0 + 10x^2z + (-10x^2z)$$
$$= 0$$

$\nabla \cdot (\nabla \times \boldsymbol{F})$ = 0

9. *(15 points)* Evaluate the line integral $\int_C (x^2+y^2)\,ds$ given C is the path given by $x=e^t \sin t$, $y=e^t \cos t$ and $0 \le t \le 3$.

$$\int_0^3 \left(e^{2t}\sin^2 t + e^{2t}\cos^2 t\right)\sqrt{(e^t\cos t + e^t\sin t)^2 + (e^t\cos t - e^t\sin t)^2}\,dt$$

$$\frac{dx}{dt} = e^t\cos t + e^t \sin t$$

$$\frac{dy}{dt} = -e^t\sin t + e^t\cos t$$

$$= \int_0^3 e^{3t}\sqrt{\cos^2 t + 2\cos t\sin t + \sin^2 t + \cos^2 t - 2\cos t \sin t + \sin^2 t}\,dt$$

$$= \int_0^3 \sqrt{2}\,e^{3t}\,dt = \frac{\sqrt{2}}{3}e^{3t}\Big|_0^3$$

$$= \frac{\sqrt{2}}{3}\left(e^9 - 1\right)$$

Answer 9: $\frac{\sqrt{2}}{3}(e^9-1)$

10. *(15 points)* Determine whether $\boldsymbol{F}(x,y,z)=(y^2\cos x+z^3)\boldsymbol{i}+(2y\sin x-4)\boldsymbol{j}+(3xz^2+2)\boldsymbol{k}$ is conservative. If so, find f such that $\boldsymbol{F}=\nabla f$. If not, state that $\boldsymbol{F}$ is not conservative.

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2+2\end{vmatrix}$$

$$=(0-0)\hat{i}+(3z^2-3z^2)\hat{j}+(2y\cos x-2y\cos x)\hat{k}$$

$$=0\hat{i}+0\hat{j}+0\hat{k}.$$

So, $\vec{F}$ is conservative.

$$\frac{\partial f}{\partial x}=y^2\cos x+z^3$$

$$f=y^2\sin x+xz^3+C(y,z)$$

$$\frac{\partial f}{\partial y}=2y\sin x+\frac{\partial C(y,z)}{\partial y}=2y\sin x-4$$

$$\Rightarrow C(y,z)=-4y+C(z)$$

$$\frac{\partial f}{\partial z}=3xz^2+C'(z)=3xz^2+2$$

$$\Rightarrow C(z)=2z+C$$

So, $f(x,y,z)=y^2\sin x+xz^3-4y+2z+C$

Conservative: True or False (circle one)

If conservative, $f=$ $y^2\sin x+xz^3-4y+2z+C$

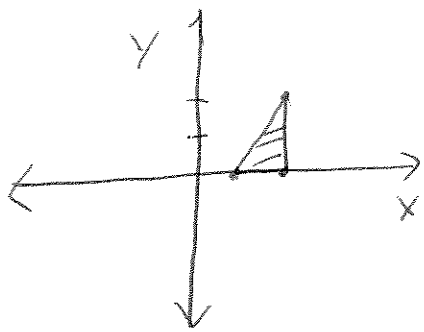
11. *(15 points)* Use Green's Theorem to evaluate $\oint_C (2xy)\,dx + (e^x + x^2)\,dy$ where C is the boundary of the triangle with vertices (1, 0), (2, 0) and (2, 2) oriented counter-clockwise.

$$M(x,y) = 2xy \qquad N(x,y) = e^x + x^2$$

$$\frac{\partial N}{\partial x} = e^x + 2x \qquad \frac{\partial M}{\partial y} = 2x$$

$$\oint_C (2xy)dx + (e^x + x^2)dy = \iint_S (e^x + 2x - 2x)\,dA$$

$$= \iint_S e^x\,dA$$



$$= \int_1^2 \int_0^{2x-2} e^x\,dy\,dx$$

$$= \int_1^2 \left(e^x y \Big|_0^{2x-2}\right) dx$$

$$= \int_1^2 (2x-2)e^x\,dx$$

$$= 2xe^x - 4e^x \Big|_1^2$$

$$= (4e^2 - 4e^2) - (2e - 4e)$$

$$= 2e$$

Answer 11: $2e$

12. *(15 points)* Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{4}} \int_{\cos(4z)}^{0} \int_0^{4yz} \cos\left(\frac{x}{y}\right) dx\, dy\, dz$.

$$= \int_0^{\pi/4} \int_{\cos(4z)}^{0} y \sin\left(\frac{x}{y}\right) \Big|_0^{4yz}$$

$$= \int_0^{\pi/4} \int_{\cos(4z)}^{0} y \sin(4z)\, dy\, dz$$

$$= \int_0^{\pi/4} \frac{y^2}{2} \sin(4z) \Big|_{\cos(4z)}^{0} dz$$

$$= \int_0^{\pi/4} -\frac{\cos^2(4z)}{2} \sin(4z)\, dz$$

$$u = \cos(4z)$$

$$du = -4\sin(4z)\, dz$$

$$\Rightarrow \frac{du}{4} = -\sin(4z)\, dz$$

$$= \int_{0=z}^{\pi/4=z} \frac{u^2}{8}\, du = \int_1^{-1} \frac{u^2}{8}\, du$$

$$= \frac{u^3}{24} \Big|_1^{-1} = -\frac{1}{24} - \frac{1}{24} = -\frac{1}{12}$$

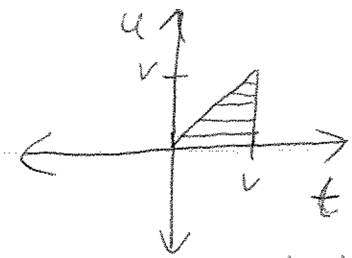
Answer 12: $\boxed{-1/12}$

Extra Credit:

(10 points) Prove the identity:

$$\int_0^x \int_0^v \int_0^u f(t)\,dt\,du\,dv = (1/2)\int_0^x (x-t)^2 f(t)\,dt$$

If we first evaluate the two inner integrals we see that the region of integration is:



$$\int_0^v \int_0^u f(t)\,dt\,du = \int_0^v \int_t^v f(t)\,du\,dt = \int_0^v u f(t)\Big|_t^v dt$$

$$= \int_0^v (v-t) f(t)\,dt$$

So, $\int_0^x \int_0^v \int_0^u f(t)\,dt\,du\,dv = \int_0^x \int_0^v (v-t) f(t)\,dt\,dv$

Now, $\int_0^x \int_0^v (v-t) f(t)\,dt\,dv$ has the same form,

so $= \int_0^x \int_t^x (v-t) f(t)\,dv\,dt$

$$= \int_0^x \frac{(v-t)^2}{2} f(t)\Big|_t^x dt$$

$$= \frac{1}{2}\int_0^x (x-t)^2 f(t)\,dt$$

$\boxed{\text{Q.E.D.}}$

Answer : ____________________

Extra Credit 2:

(5 points) What is 5+2?

$\boxed{7}$