Math 2210 - Quiz 2

University of Utah

Summer 2007

Name: Solutions

1. (10 points)

   Calculate the following double integrals.

   (a) (3 points)
   $\int_{-1}^{4}\int_{1}^{2}(x+y^2)dydx$

   Solution

$$\int_{-1}^{4}\int_{1}^{2}(x+y^2)dydx = \int_{-1}^{4} xy + \frac{y^3}{3}|_1^2 dx$$
$$= \int_{-1}^{4} x + \frac{7}{3}dx = \frac{x^2}{2} + \frac{7x}{3}|_{-1}^{4} = (8+\frac{28}{3}) - (\frac{1}{2}-\frac{7}{3}) = \frac{115}{6}$$

   (b) (3 points)
   $\int_{\frac{1}{2}}^{1}\int_{0}^{2x}\cos{(\pi x^2)}dydx$

   Solution

$$\int_{\frac{1}{2}}^{1}\int_{0}^{2x}\cos{(\pi x^2)}dydx = \int_{\frac{1}{2}}^{1} y\cos{(\pi x^2)}|_0^{2x} = \int_{\frac{1}{2}}^{1} 2x\cos{(\pi x^2)}dx$$
$$= \frac{\sin{(\pi x^2)}}{\pi}|_{\frac{1}{2}}^{1} = \frac{\sin{(\pi)} - \sin{(\frac{\pi}{4})}}{\pi} = -\frac{1}{\sqrt{2}\pi}$$

(c) (4 points)

$$\int_0^{\pi}\int_0^{1-\cos\theta} r\sin\theta dr d\theta$$

Solution

$$\int_0^{\pi}\int_0^{1-\cos\theta} r\sin\theta dr d\theta = \int_0^{\pi} \frac{r^2\sin\theta}{2}|_0^{1-cos\theta} d\theta = \int_0^{\pi} \frac{(1-\cos\theta)^2\sin\theta}{2} d\theta$$
$$= \frac{(1-\cos\theta)^3}{6}|_0^{\pi} = \frac{2^3}{6} - \frac{0^3}{6} = \frac{8}{6} = \frac{4}{3}$$

Note - In the second integral we used the substitution $u = x^2$, and for the third integral we used the substitution $u = 1 - \cos\theta$.

2. (10 points)

   Evaluate the given integrals using either cartesian or polar integration, whichever works best for the given problem, and sketch the domain of integration. (Hint: Make the sketch first.)

   (a) (5 points)
   $\int\int_S (x^2 - xy)dA$;
   S is the region between $y = x$ and $y = 3x - x^2$.

   Solution

   First we calculate the limits on the x values, which will be the x values where the two curves intersect. These are:

$$x = 3x - x^2 \rightarrow 2x = x^2 \rightarrow x = \{0, 2\}$$

   So, our integral will be:

$$\int_0^2 \int_x^{3x-x^2} (x^2 - xy)dydx$$

   Which, when we evaluate it we get:

$$\int_0^2 \int_x^{3x-x^2} (x^2 - xy)dydx = \int_0^2 x^2y - \frac{xy^2}{2}|_x^{3x-x^3} dx$$
$$= \int_0^2 (x^2(3x - x^2) - \frac{x(3x - x^2)^2}{2}) - (x^3 - \frac{x^3}{2})dx =$$
$$\int_0^2 (2x^4 - \frac{x^5}{2} - 2x^3)dx = \frac{2x^5}{5} - \frac{x^6}{12} - \frac{x^4}{2}|_0^2$$
$$= \frac{64}{5} - \frac{64}{12} - 8 = \frac{192 - 80 - 120}{15} = -\frac{8}{15}$$

(b) (5 points)

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \sin{(x^2+y^2)} dx dy$$

Solution

As the sketch of the domain above indicates, we're going to want to integrate this in polar coordinates. So, switching to polar coordinates are integral becomes:

$$\int_0^{\frac{\pi}{2}} \int_0^1 \sin{(r^2)} r dr d\theta$$

We can evaluate this if we make the substitution $u = r^2$ and so $du = 2rdr$. Under this substitution the integral becomes:

$$\int_0^{\frac{\pi}{2}} \int_0^1 \frac{\sin u}{2} du d\theta = \int_0^{\frac{\pi}{2}} -\frac{\cos u}{2}|_0^1 d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{1-\cos{(1)}}{2} d\theta = \frac{1-\cos{(1)}}{2}(\frac{\pi}{2}-0) = \pi(\frac{1-\cos{(1)}}{4})$$

3. (10 points)

For the solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 4$ calculate:

(a) (3 points)

The volume of the solid.

Solution

We can figure this out using either high school geometry or we can take the triple integral:

$$\int_0^{2\pi}\int_0^3\int_0^4 r dz dr d\theta = \int_0^{2\pi}\int_0^3 4r dr d\theta = \int_0^{2\pi} 2r^2|_0^3 d\theta$$
$$= \int_0^{2\pi} 18 d\theta = 36\pi$$

(b) (3 points)

The mass of the solid assuming the density is given by:

$$\rho(x, y, z) = 2(x^2 + y^2 + z^2)$$

Solution

The limits of integration here will be exactly the same. The only different here is that our integrand will be $2(x^2 + y^2 + z^2) = 2(r^2 + z^2)$ instead of 1. Our integral will therefore be:

$$\int_0^{2\pi}\int_0^3\int_0^4 2(r^2 + z^2) r dz dr d\theta = \int_0^{2\pi}\int_0^3 (2zr^3 + \frac{2z^3 r}{3})|_0^4 dr d\theta$$
$$= \int_0^{2\pi}\int_0^3 (8r^3 + \frac{128r}{3}) dr d\theta$$

$$\int_0^{2\pi} (\frac{8r^4}{4} + \frac{64r^2}{3})|_0^3 d\theta = \int_0^{2\pi} (162 + 192) d\theta = 2\pi(354) = 708\pi$$

(c) (4 points)

The center of mass of the solid with the above density function. (Note: You should only have to calculate one of the three center of mass coordinates here. The other two should be obvious, but you should say *why* they're obvious.)

Solution

Given both the solid and the density function are symmetric around the z-axis, both the x and y coordinates of the center of mass will be 0, as the center of mass will have to be on the z-axis. The z-coordinate of the center of mass can be calculated by calculating the moment M_{xy} of the solid:

$$M_{xy} = \int_0^{2\pi} \int_0^3 \int_0^4 2(r^2 + z^2) zr dz dr d\theta = \int_0^{2\pi} \int_0^3 (z^2 r^3 + \frac{z^4 r}{2}) dr d\theta$$
$$= \int_0^{2\pi} \int_0^3 (\frac{z^2 r^4}{4} + \frac{z^4 r^2}{4})|_0^4 dr d\theta = \int_0^{2\pi} \int_0^3 (4r^4 + 64r^2) dr d\theta =$$
$$\int_0^{2\pi} (4(81) + 64(9)) d\theta = 2\pi(900) = 1800\pi$$

And so therefore the z-coordinate of the center of mass will be:

$$\overline{z} = \frac{1800\pi}{708\pi} \approx 2.54$$

And so the center of mass will be at: $(0, 0, 2.54)$.

4. (10 points)

Calculate the following quantities:

(a) (5 points)

The surface area of the part of the surface $z = \frac{x^2}{4} + 4$ that is cut off by the planes $x = 0$, $x = 1$, $y = 0$, and $y = 2$.

Solution

The partial derivatives of this function $z = f(x, y)$ with respect to x and y are, respectively:

$$\frac{\partial z}{\partial x} = \frac{x}{2}$$

and

$$\frac{\partial z}{\partial y} = 0$$

Therefore, the surface area of this region will be:

$$\int_0^1 \int_0^2 \sqrt{1 + (\frac{x}{2})^2 + 0^2} dy dx = \int_0^1 2\sqrt{1 + \frac{x^2}{4}} = \int_0^1 \sqrt{4 + x^2} dx$$
$$= \frac{\sqrt{5}}{2} + 2 \ln \frac{1 + \sqrt{5}}{2}$$

Note - I'm very sorry about this problem. Its solution required the use of a table of integrals, or the knowledge of how to do a very difficult integral. I wouldn't have been able to do it without a table of integrals. I should have given you the integral on the exam. When I did this problem before I made a mistake and thought it was easier than it is. So, I'm giving everybody who attempted this problem and got it down to the hard integral, but was unable to do it, full credit.

(b) (5 points)
The area in the first quadrant between the curves defined by the equations $x^2 + y^2 = 36$ and $x^2 - 6x + y^2 = 0$. (Note - You should do this problem as a double integral in polar coordinates. Any other argument will not receive full credit.)

Solution

This is a webworks problem and a problem I did in class. If you graph the region out you get the area pictured below:

If we represent these two curves in polar coordinates we find that the first has the polar equation:

$$r = 6$$

While the second equation has the form:

$$r^2 - 6r\cos\theta = 0 \rightarrow r = 6\cos\theta$$

Given we're dealing with all the points in the first quadrant our integral will go from $0 \leq \theta \leq \frac{\pi}{2}$, and so our double integral for this area will be:

$$\int_0^{\frac{\pi}{2}} \int_{6\cos\theta}^{6} r dr d\theta = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}|_{6\cos\theta}^{6} d\theta = \int_0^{\frac{\pi}{2}} (18 - 18\cos^2\theta) d\theta$$
$$= \int_0^{\frac{\pi}{2}} 18\sin^2\theta d\theta = \int_0^{\frac{\pi}{2}} 9(1 - \cos(2\theta)) d\theta = (9\theta - \frac{9\sin(2\theta)}{2})|_0^{\frac{\pi}{2}} = \frac{9\pi}{2}$$

Note that this also could have been deduced by looking at the picture and using a clever application of high school geometry. However, as the problem asks you to figure out the area using a polar integral, such an approach would not have received full credit.

5. (10 points)

Evaluate the integral $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$.

(Note - You must provide a formal evaluation of any integral. For full credit you can't just quote a result from the textbook or from lecture, you must rederive the result. Also, your final answer may be in terms of μ and σ.)

Solution

If we begin with the substitution $u = \frac{(x-\mu)}{\sqrt{2}\sigma}$ then we have $du = \frac{dx}{\sqrt{2}\sigma}$ and the integral becomes:

$$\int_{-\infty}^{\infty} \sqrt{2}\sigma e^{-u^2} du = \sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-u^2} du$$

Now, this integral is tricky, but we went over it in class, in the review session, and it's covered in the textbook. So, here's how it's done. First, we take a look at the double integral:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = [\int_{-\infty}^{\infty} e^{-x^2} dx]^2$$

So, if we can calculate the double integral above, its square root will be the integral for which we're searching. The trick here is to convert the above double integral into a polar integral over the entire plane:

$$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = \int_0^{2\pi} \int_0^{\infty} \frac{e^{-u}}{2} du d\theta$$

where here we used the substitution $u = r^2$ and so $du = 2rdr$

$$\int_0^{2\pi} -\frac{e^{-u}}{2}|_0^{\infty} du d\theta = \int_0^{2\pi} (-0 - (-\frac{1}{2})) d\theta = 2\pi(\frac{1}{2}) = \pi$$

So, the double integral above is equal to π, and therefore the single integral for which we're looking is equal to $\sqrt{\pi}$. Using this we get:

$$\sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{2\pi}\sigma$$