Math 2210 - Quiz 1

University of Utah

Summer 2007

Name: Quiz Solutions

1. (10 points) For the vectors **a** = $(3,4,7)$ and **b** = $(2,1,3)$ calculate

   (a) (1 point)
   $\mathbf{a}+\mathbf{b}$

   $(3,4,7)+(2,1,3)=(5,5,10)$

   (b) (1 point)
   $\mathbf{a}-\mathbf{b}$

   $(3,4,7)-(2,1,3)=(1,3,4)$

   (c) (2 points)
   $\mathbf{a}\cdot\mathbf{b}$

   $(3,4,7)\cdot(2,1,3)=3(2)+1(4)+7(3)=31$

(d) (2 points)
a $\times$ **b**

$(3,4,7)\times(2,1,3) = (4(3)-7(1), 7(2)-3(3), 3(1)-4(2)) = (5,5,-5)$

(e) (1 point)
Explain why $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{b}$ doesn't make sense.

$\mathbf{a}\cdot\mathbf{b}$ produces a scalar, and you cannot take the cross product of a vector with a scalar.

(f) (3 points)
The equation for the plane that contains vectors **a** and **b** that passes through the point $(1,4,2)$.
Note - Equation should be in the form $Ax + By + Cz = D$.

We calculated the normal vector to the plane when we calculated the cross product above. So, we know that $(A,B,C) = (5,5,-5)$. We just need to calculate the value for D, and that we get by requiring that $(x,y,z) = (1,4,2)$ satisfies the equeation. Plugging in these values we get:

$$5(1) + 5(4) - 5(2) = 15 = D$$

So, the equation for our plane is:

$$5x + 5y - 5z = 15$$

Note that this could be simplified to:

$$x + y - z = 3$$

Either of the above two equations would be acceptable.

2. (10 points)

For the vector valued function $\mathbf{r}(t) = (t^2, \frac{4\sqrt{3}}{3}t^{\frac{3}{2}}, 3t)$ calculate:

(a) (3 points)
The derivative $\mathbf{r}^{'}(t)$

$$\mathbf{r}^{'}(t) = (2t, 2\sqrt{3}t^{\frac{1}{2}}, 3)$$

(b) (3 points)
The second derivative $\mathbf{r}^{''}(t)$

$$\mathbf{r}^{''}(t) = (2, \frac{\sqrt{3}}{t^{\frac{1}{2}}}, 0)$$

(c) (4 points)
The arc length of the curve $\mathbf{r}(t)$ from $t = 1$ to $t = 4$.

$$\begin{gathered} L = \int_1^4 ||\mathbf{r}^{'}(t)||dt = \int_1^4 \sqrt{(2t)^2 + (2\sqrt{3}t^{\frac{1}{2}})^2 + 3^2}dt \\ = \int_1^4 \sqrt{4t^2 + 12t + 9}dt = \int_1^4 \sqrt{(2t+3)^2}dt \\ = \int_1^4 (2t+3)dt = t^2 + 3t|_1^4 = (4^2 + 3(4)) - (1^2 + 3) \\ = 28 - 4 = \mathbf{24} \end{gathered}$$

3. (10 points)

(a) (6 points)

Find the symmetric equations of the tangent line to the curve $\mathbf{r}(t) = (2t^2, 4t, t^3)$ at $t = 1$.

The derivative of the curve is given by:

$$\mathbf{r}^{'} = (4t, 4, 3t^2)$$

The tangent line at $t = 1$ is given by $\mathbf{r}^{'}(1) = (4, 4, 3)$ and it has as its initial point $\mathbf{r}(1) = (2, 4, 1)$. So, the symmetric equations for this line are:

$$\frac{x-2}{4} = \frac{y-4}{4} = \frac{z-1}{3}$$

(b) (4 points)

Calculate the distance between the above tangent line and the point $(4, 3, 1)$.

There are two ways to do this problem:

Method 1

Parameterize the line with a variable t, so that we have:

$$x(t) = 4t + 2, y(t) = 4t + 4, z(t) = 3t + 1$$

and as t changes and we move along the line, connect a vector from the given position on the line to the point $(4, 3, 1)$. The paramterized equation for this vector will be

$$\mathbf{u}(t) = P - L(t) = (2 - 4t, -1 - 4t, -3t)$$

Now, the distance will be shortest when this vector is perpendicular to the line, so we need to find where this vector has a dot product of 0 with the line's vector:

$$0 = \mathbf{u}(t) \cdot \mathbf{v} = 4(2 - 4t) + 4(-1 - 4t) + 3(-3t) = 4 - 41t$$
$$\rightarrow t = \frac{4}{41}$$

If we plug in this value for t in the equation for our line we get the point $(\frac{98}{41}, \frac{180}{41}, \frac{53}{41})$ which has a distance from the point $(4, 3, 1)$ equal to:

$$d = \sqrt{(4 - \frac{98}{41})^2 + (3 - \frac{180}{41})^2 + (1 - \frac{53}{41})^2} = 2.147$$

Method 2

We can take a vector pointing from our starting point of $(2, 4, 1)$ to the point $(4, 3, 1)$ and break it up into its components parallel to and perpendicular the vector for our line $\mathbf{v} = (4, 4, 3)$. The length of the perpendicular component will be the distance.

To calculate the parellel component we project the vector $\mathbf{w} = (4, 3, 1) - (2, 4, 1) = (2, -1, 0)$ onto the vector $\mathbf{u} = (4, 4, 3)$ to get:

$$proj_{\mathbf{u}}\mathbf{w} = (\frac{\mathbf{w} \cdot \mathbf{u}}{||\mathbf{u}||^2})\mathbf{u} = \frac{8 - 4 + 0}{16 + 16 + 9}(4, 4, 3)$$
$$= \tfrac{4}{41}(4, 4, 3) = (\tfrac{16}{41}, \tfrac{16}{41}, \tfrac{12}{41})$$

And so, the perpendicular component of this decomposition would be:

$$perp_{\mathbf{u}}\mathbf{w} = \mathbf{w} - proj_{\mathbf{u}}\mathbf{w} = (2 - \frac{16}{41}, -1 - \frac{16}{41}, -\frac{12}{41})$$

So, the distance is:

$$||perp_{\mathbf{u}}\mathbf{w}|| = \sqrt{(2 - \frac{16}{41})^2 + (-1 - \frac{16}{41})^2 + (-\frac{12}{41})^2} = 2.147$$

So, both methods work and return the same answer, as they must if they're both right.

Note - This is without question the hardest problem on the quiz. I assigned it because it was also a webworks problem and wanted to see who could repeat the method on the quiz. However, don't feel bad if you got it wrong. Only 3 people did this correctly.

4. (10 points)

Convert the following points from Cartesian to cylindrical coordinates:

(a) (1 point)
$(5, 2, 7)$

$$r = \sqrt{x^2 + y^2} = \sqrt{5^2 + 2^2} = \sqrt{29}$$
$$\theta = \arctan\frac{y}{x} = \arctan\frac{2}{5} = .3805rad = 21.8^\circ$$
$$z = z = 7$$

(b) (1 point)
$(2, 4, 9)$

$$r = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$$
$$\theta = \arctan\frac{4}{2} = 1.1072rad = 63.4^\circ$$
$$z = z = 9$$

Convert the following points from Cartesian to spherical coordinates:

(a) (1 point)
$(5, 2, 7)$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{5^2 + 2^2 + 7^2} = \sqrt{78}$$
$$\theta = \arctan\frac{y}{x} = \arctan\frac{2}{5} = .3805rad = 21.8^\circ$$
$$\phi = \arccos\frac{z}{\rho} = \arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}} = \arccos\frac{7}{\sqrt{78}} = .6557rad = 37.6^\circ$$

(b) (1 point)
$(4, 11, 3)$

$$\rho = \sqrt{4^2 + 11^2 + 3^2} = \sqrt{146}$$
$$\theta = \arctan\frac{11}{4} = 1.222rad = 70.0°$$
$$\phi = \arccos\frac{3}{\sqrt{146}} = 1.320rad = 75.6°$$

Convert the following from spherical to cylindrical coordinates:

(a) (1 point)
$(5, \frac{\pi}{3}, \frac{\pi}{6})$

$$r = \rho\sin\phi = 5\sin\frac{\pi}{6} = \frac{5}{2}$$
$$\theta = \theta = \frac{\pi}{3}$$
$$z = \rho\cos\phi = 5\cos\frac{\pi}{6} = \frac{5\sqrt{3}}{2}$$

(b) (1 point)
$(2, 0, \pi)$
$r = 2\sin\pi = 0$
$\theta = 0$
$z = 2\cos\pi = -2$

Convert the following from cylindrical to Cartesian coordinates:

(a) (1 point)
$(5, \frac{\pi}{6}, -2)$

$$x = r\cos\theta = 5\cos\frac{\pi}{6} = \frac{5\sqrt{3}}{2}$$
$$y = r\sin\theta = 5\sin\frac{\pi}{6} = \frac{5}{2}$$
$$z = z = -2$$

(b) (1 point)
$(3, \frac{4\pi}{3}, -7)$

$$x = 3\cos\frac{4\pi}{3} = -\frac{3}{2}$$
$$y = 3\sin\frac{4\pi}{3} = -\frac{3\sqrt{3}}{2}$$
$$z = -7$$

(2 points)
Convert the equation $x^2 + y^2 = 9$ to spherical coordinates.

Given in spherical coordinates we have the relations:

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$

Plugging these relations into $x^2 + y^2 = 9$ we get:

$$\rho^2\cos\theta^2\sin\phi^2 + \rho^2\sin\theta^2\sin\phi^2 = 9$$
$$\rightarrow \rho^2\sin\phi^2 = 9$$

5. (10 points)

For each equation below match it up with its corresponding surface and give the name of the surface. (So, where it asks for surface, give the letter of the surface corresponding to the equation.)

(a) (2 points)
$4x^2 + 16y^2 - 32z = 0$

Surface: A

Surface Description: Elliptic Paraboloid

(b) (2 points)
$x^2 - y^2 + z = 0$

Surface: C

Surface Description: Hyperbolic Paraboloid (Monkey Saddle)

(c) (2 points)
$9x^2 + 25y^2 + 9z^2 = 225$

Surface: F

Surface Description: Ellipsoid

(d) (2 points)
$9x^2 - z^2 + 9y^2 - 9 = 0$

Surface: E
Surface Description: Hyperboloid of One Sheet

(e) (2 points)
$x^2 + y^2 - 4z^2 = 0$

Surface: D
Surface Description: Elliptic Cone

The possible surface descriptions are: ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic paraboloid, hyperbolic paraboloid, and elliptic cone.

The surface drawings are on the next page.