Math 2210 - Midterm

University of Utah

Summer 2007

Name: Midterm Solutions

50 points possible.

1. (10 points) *Partial Derivatives*

   A function of two variables that satisfies Laplace's equation:

   $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

   is said to be harmonic. Show that the functions defined below are harmonic functions.

   (a) (3 points)

   $$f(x,y) = x^3y - xy^3$$

   Solution

   $f_x(x,y) = 3x^2y - y^3$ $\qquad$ $f_y(x,y) = x^3 = 3xy^2$

   $f_{xx}(x,y) = 6xy$ $\qquad$ $f_{yy}(x,y) = -6xy$

   Therefore, $f_{xx}(x,y) + f_{yy}(x,y) = 6xy - 6xy = 0$

(b) (3 points)

$$f(x, y) = e^x \sin y$$

Solution

$f_x(x, y) = e^x \sin y$ $\qquad$ $f_y = e^x \cos y$

$f_{xx}(x, y) = e^x \sin y$ $\qquad$ $f_{yy}(x, y) = -e^x \sin y$

Therefore, $f_{xx}(x, y) + f_{yy}(x, y) = e^x(\sin y - \sin y) = 0$

(c) (4 points)

$$f(x, y) = \ln(4x^2 + 4y^2)$$

Solution

$f_x(x, y) = \dfrac{2x}{x^2 + y^2}$ $\qquad$ $f_y(x, y) = \dfrac{2y}{x^2 + y^2}$

$f_{xx}(x, y) = \dfrac{2y^2 - 2x^2}{(x^2 + y^2)^2}$ $\qquad$ $f_{yy}(x, y) = \dfrac{2x^2 - 2y^2}{(x^2 + y^2)^2}$

Therefore, $f_{xx}(x, y) + f_{yy}(x, y) = \dfrac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \dfrac{2x^2 - 2y^2}{(x^2 + y^2)^2} = 0$

2. (10 points) *Limits*

Determine each of the following limits, or state it does not exist and give an explanation as to why.

(a) (3 points)

$$\lim_{(x,y)\to(1,1)} (x^2 + y^2 + \cos(xy - 1))$$

Solution

If we just plug in the value $(1, 1)$ into the equation we get:

$$1^2 + 1^2 + \cos(1 * 1 - 1) = 1 + 1 + 1 = 3$$

As this is a sum of elementary functions it is continuous, and so this is its limit.

(b) (3 points)

$$\lim_{(x,y)\to(0,0)} \frac{\tan((x^2 + y^2)^2)}{(x^2 + y^2)^3}$$

Solution

If we use the substitution $r = x^2 + y^2$ and convert to polar coordinates the above limit becomes:

$$\lim_{r\to 0} \frac{\tan r^4}{r^6} = \frac{0}{0}$$

Now, if we apply L'Hospital's rule here we get:

$$\lim_{r\to 0} \frac{4r^3 \sec^2 r^4}{6r^5} = \frac{2\sec^2 r^4}{3r^2} = \frac{2}{0}$$

So, the limit is divergent and does not exist.

(c) (4 points)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Solution

Again, if we convert this to polar coordinates we get:

$$\lim_{(x,y)\to 0} \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta$$

As this limit is different for different values of θ there is not one unique limit, and so the limit does not exist.

3. (10 points) *Gradients*

Calculate the gradients of the following functions:

(a) (2 points)

$$f(x, y) = e^{-xy}$$

Solution

$$\bigtriangledown f(x, y) = (f_x, f_y) = (-ye^{-xy}, -xe^{-xy}) = -ye^{-xy}\mathbf{i} - xe^{-xy}\mathbf{j}$$

(b) (2 points)

$$f(x, y, z) = x^3 y - y^2 z^2$$

Solution

$$\begin{aligned} \bigtriangledown f(x, y, z) &= (f_x, f_y, f_z) = (3x^2 y, x^3 - 2yz^2, -2y^2 z) \\ &= 3x^2 y\mathbf{i} + (x^3 - 2yz^2)\mathbf{j} - 2y^2 z\mathbf{k} \end{aligned}$$

(2 points)

For the function of two variables above, calculate the directional derivative at the point $(1, -1)$ in the direction of $\mathbf{v} = -\mathbf{i} + \sqrt{3}\mathbf{j}$.

Solution

The directional derivative of a function at a point is just the gradient of the function at that point multiplied by a unit vector in the given direction.

Here, our direction vector is $\mathbf{v} = -\mathbf{i} + \sqrt{3}\mathbf{j}$, and a unit vector in the same direction is:

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{(-1, \sqrt{3})}{\sqrt{(-1)^2 + \sqrt{3}^2}} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{J}$$

Now, the value of the gradient at the point $(1, -1)$ is:

$$\triangledown f(1, -1) = -(-1)e^{-1*(-1)}\mathbf{i} - 1e^{-1*(-1)}\mathbf{j} = e\mathbf{i} - e\mathbf{j} = (e, -e)$$

The directional derivative is given by:

$$D_{\mathbf{u}} f(1, -1) = \triangledown f(1, -1) \cdot \mathbf{u} = (e, -e) \cdot (-\frac{1}{2}, \frac{\sqrt{3}}{2})$$
$$= \frac{-e - e\sqrt{3}}{2} = -\frac{e(1 + \sqrt{3})}{2}$$

(2 points)

For the function of three variables above, calculate the directional derivative at the point $(-2, 1, 3)$ in the direction of $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

Solution

This problem is similar to the one right before. We figure out the gradient at the given point, find a unit vector in the same direction as **v**, and then take the dot product of the gradient at that point with the unit vector.

The gradient of the function at the point $(-2, 1, 3)$ is:

$$\triangledown f(-2, 1, 3) = (3 * (-2)^2 * 1, (-2)^3 - 2(1)(3)^2, -2(3)(1)^2)$$
$$= (12, -26, -6)$$

A unit vector **u** in the same direction as **v** is:

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{(1,-2,2)}{\sqrt{(1^2+(-2)^2+2^2)}} = (\frac{1}{3}, \frac{-2}{3}, \frac{2}{3})$$

Now, the directional derivative is given by:

$$D_{\mathbf{u}}f(-2,1,3) = \bigtriangledown f(-2,1,3) \cdot \mathbf{u}$$
$$= (12,-26,-6) \cdot (\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}) = \frac{52}{3}$$

(2 points)

For the function of three variables above at the point $(-2,1,3)$ provide the maximum value of the directional derivative and provide a unit vector in this maximal direction.

Solution

The direction that provides the maximal directional derivative is the direction of the gradient, and that maximal value is the length of the gradient.

So, for our problem the maximal directional derivative has magnitude:

$$|| \bigtriangledown f(-2,1,3)|| = \sqrt{12^2+(-26)^2+(-6)^2} = \sqrt{856}$$

And so, the unit vector in this maximal direction will be:

$$\mathbf{u} = \frac{\bigtriangledown f(-2,1,3)}{|| \bigtriangledown f(-2,1,3)||} = (\frac{12}{\sqrt{856}}, \frac{-26}{\sqrt{856}}, \frac{-6}{\sqrt{856}})$$

4. (10 points) *Extrema and Tangent Planes*

For the surface defined by:

$$z = x^3 + y^3 - 6xy$$

(a) (6 points)

Determine all the critical points of the function, and determine if these points are local maxima, local minima, or saddle points.

Solution

Because the function is a polynomial and we're allowing the entire plane as our space of possible inputs, the only possible critical points are the points at which the partial derivatives are simultaneously 0. The two partial derivatives for this function are:

$$\frac{\partial z}{\partial x} = 3x^2 - 6y$$
$$\frac{\partial z}{\partial y} = 3y^2 - 6x$$

Now, if we set both of these equal to 0 simultaneously, we can use the second equation to solve for y in terms of x:

$$x = \frac{3y^2}{6} = \frac{y^2}{2}$$

If we then plug this equation for x into our first equation and set it equal to 0, we get:

$$\frac{3y^4}{4} - 6y = 0$$

This equation has solutions:

$y = 0$ and $y = 2$

With corresponding x solutions:

$x = 0$ and $x = 2$

So, the two critical points for this function are:

$(0, 0)$ and $(2, 2)$.

Now, to check if these points are local maxima, local minima, or saddle points we check the value of:

$$D = f_{xx}f_{yy} - f_{xy}^2$$

For our equation we have:

$$f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = f_{yx} = -6$$

So, for our two critical points we have:

$$D(0, 0) = (0)(0) - (-6)^2 = -36$$

So, it's a saddle point, as D is negative.

$$D(2, 2) = (12)(12) - (-6)^2 = 108$$

So, it's a minimum as D is positive and f_{xx} is as well.

(b) (4 points)
Calculate the equation for the tangent plane to the function at the point $(3,3,0)$.
Note - The equation for the plane should be of the form

$$Ax + By + Cz = D.$$

Solution

For $x = 3$ and $y = 3$ we have for our equation:

$$f_x(3,3) = 3(3)^2 - 6(3) = 9$$
$$\text{and}$$
$$f_y(3,3) = 3(3)^2 - 6(3) = 9$$

This means that our tangent plane will have the equation:

$$9x + 9y - z = D$$

Now, our tangent plane must coincide with the surface at this point, and the surface at this point has z value 0. So, $(x, y, z) = (3, 3, 0)$ must be a point on our plane, and so D must be:

$$D = 9(3) + 9(3) - 0 = 54$$

Therefore, the equation for our tangent plane is:

$$9x + 9y - z = 54$$

5. (5 points) *Chain Rule*

   For the implicitly defined curve:

$$x^2 \cos y - y^2 \sin x = 0$$

   calculate $\frac{dy}{dx}$.

   Solution

   To solve this we just use the implicit function theorem. Using the implicit equation:

$$F(x, y) = x^2 \cos y - y^2 \sin x = 0$$

   We get:

$$\frac{\partial F}{\partial x} = 2x \cos y - y^2 \cos x$$

$$\text{and}$$

$$\frac{\partial F}{\partial y} = -x^2 \sin y - 2y \sin x$$

   Now, according to the implicit function theorem we have:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{2x \cos y - y^2 \cos x}{x^2 \sin y + 2y \sin x}$$

6. (5 points) *Chapter 11*

Find the symmetric equations of the tangent line to the curve

$$\mathbf{r}(t) = (2t^2, 4t, t^3)$$

at $t = 1$.

Solution

This is just problem 3a) from the first quiz. The solution is:

The derivative of the curve is given by:

$$\mathbf{r}^{'}(t) = (4t, 4, 3t^2)$$

The tangent line at $t = 1$ is given by $\mathbf{r}^{'}(1) = (4, 4, 3)$ and it has as its initial point $\mathbf{r}(1) = (2, 4, 1)$. So, the symmetric equations for this line are:

$$\frac{x-2}{4} = \frac{y-4}{4} = \frac{z-1}{3}$$