Math 2210 - Final Exam

University of Utah

Summer 2007

Name: Solutions

1. (10 points)
   For the vectors:

$$\mathbf{a} = -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$
and
$$\mathbf{b} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}.$$

Calculate:

(a) (2 points)
$\mathbf{a} + \mathbf{b}$

Solution

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (-3 + (-1))\mathbf{i} + (2 + 2)\mathbf{j} + (-2 + (-4))\mathbf{k} \\ &= -4\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}\end{aligned}$$

(b) (2 points)
$\mathbf{a} - \mathbf{b}$

Solution

$$\begin{aligned}\mathbf{a} - \mathbf{b} &= (-3 - (-1))\mathbf{i} + (2 - 2)\mathbf{j} + (-2 - (-4))\mathbf{k} \\ &= -2\mathbf{i} + 0\mathbf{j} + 2\mathbf{k}\end{aligned}$$

(c) (3 points)
$\mathbf{a} \cdot \mathbf{b}$

Solution

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (-3)(-1) + (2)(2) + (-2)(-4) \\ &= 3 + 4 + 8 = 15\end{aligned}$$

(d) (3 points)
$\mathbf{a} \times \mathbf{b}$

Solution

$$\begin{gathered}\mathbf{a} \times \mathbf{b} \\ = ((2)(-4) - (-2)(2))\mathbf{i} + ((-2)(-1) - (-3)(-4))\mathbf{j} \\ +((-3)(2) - (2)(-1))\mathbf{k} \\ = -4\mathbf{i} - 10\mathbf{k} - 4\mathbf{k}\end{gathered}$$

2. (10 points)

For the function:

$$f(x, y, z) = x^3y + y^2z^2$$

Calculate:

(a) (4 points)

The gradient $\bigtriangledown f(x, y, z)$ of the function $f(x, y, z)$.

Solution

$$\bigtriangledown f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

So,

$$\bigtriangledown f(x, y, z) = (3x^2y)\mathbf{i} + (x^3 + 2yz^2)\mathbf{j} + (2y^2z)\mathbf{k}$$

(b) (4 points)

The directional derivative at the point $\mathbf{p} = (-2, 1, 3)$ in the direction of the vector $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

Solution

First we must figure out the unit vector in the direction of **a**. We will denote this vector by **u**:

$$\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{\mathbf{a}}{\sqrt{1^2 + (-2)^2 + 2^2}} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

The gradient of our function at the point $(-2, 1, 3)$ will be:

$$\bigtriangledown f(-2, 1, 3) = 3(-2)^2(1)\mathbf{i} + ((-2)^3 + 2(1)(3)^2)\mathbf{j} + (2(1)^2(3))\mathbf{k}$$
$$= 12\mathbf{i} + 10\mathbf{j} + 6\mathbf{k}$$

So, the directional derivative will be:

$$D_{\mathbf{u}}(f(\mathbf{p})) = \bigtriangledown f(-2,1,3) \cdot \mathbf{u} = (12)(\frac{1}{3}) + (10)(-\frac{2}{3}) + (6)(\frac{2}{3}) = \frac{4}{3}.$$

(c) (2 points)
The maximum value of the directional derivative at the point $\mathbf{p} = (-2,1,3)$.

Solution

The maximum value of the directional derivative at $\mathbf{p}$ is just the magnitude of the gradient at that point:

$$|| \bigtriangledown f(\mathbf{p})|| = \sqrt{(12)^2 + (10)^2 + (6)^2} = 2\sqrt{70}$$

3. (10 points)

Evaluate the double integral:

$$\int\int_S e^{x^2+y^2} dA$$

Where S is the region enclosed by $x^2 + y^2 = 4$. Make a sketch of this region before you calculate the integral.

Solution

We can see that our domain of integration is best suited for polar coordinates, and if we convert the integral to polar coordinates we get:

$$\int\int_S e^{x^2+y^2} dA = \int_0^{2\pi} \int_0^2 e^{r^2} r dr d\theta$$

If we make the substitution: $u = r^2$ and so $du = 2rdr$ we get:

$$\int_0^{2\pi} \int_0^2 e^{r^2} r dr d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^4 e^u du d\theta = \frac{1}{2} \int_0^{2\pi} (e^4 - 1) d\theta$$
$$= \tfrac{1}{2}(2\pi - 0)(e^4 - 1) = \pi(e^4 - 1)$$

4. (10 points)

Evaluate the line integral:

$$\int_C xe^y ds$$

Where C is the line segment from $(-1, 2)$ to $(1, 1)$.

Note - As always, ds here means the differential with respect to arc length.

Solution

If we parameterize the line segment using linear equations like so:

$$x(t) = 2t - 1$$
$$y(t) = -t + 2$$
$$0 \leq t \leq 1$$

The we have:

$$\frac{dx}{dt} = 2$$
$$\frac{dy}{dt} = -1$$

And our line integral becomes:

$$\int_C xe^y ds = \int_0^1 x(t)e^{y(t)}\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt =$$
$$\int_0^1 (2t-1)e^{(-t+2)}\sqrt{(2)^2+(-1)^2}dt$$
$$= \sqrt{5}e^2[2\int_0^1 te^{-t}dt - \int_0^1 e^{-t}dt] = \sqrt{5}e^2[2(-te^{-t} - e^{-t}|_0^1) - (-e^{-t}|_0^1)]$$
$$\sqrt{5}e^2[2(-e^{-1}-e^{-1}-(-1))-(-e^{-1}-(-1))] = \sqrt{5}e^2[-4e^{-1}+2+e^{-1}-1]$$
$$= \sqrt{5}e^2[-3e^{-1}+1] = \sqrt{5}e(e-3)$$

5. (10 points)

Evaluate the line integral:

$$\oint_C (e^{3x} + 2y)dx + (x^2 + \sin y)dy$$

Where C is the rectangle with vertices $(2,1)$, $(6,1)$, $(6,4)$, and $(2,4)$ traversed in the counter-clockwise direction.

Hint - Use Green's theorem.

Solution

We note that, in our standard notation, our vector field here is:

$$\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} = (e^{3x} + 2y)\mathbf{i} + (x^2 + \sin y)\mathbf{j}$$

where

$$M(x,y) = e^{3x} + 2y$$

and

$$N(x,y) = x^2 + \sin y$$

and therefore:

$$\frac{\partial N}{\partial x} = 2x$$

while

$$\frac{\partial M}{\partial y} = 2$$

So, Green's theorem tells us:

$$\oint_C M dx + N dy = \int\int_S (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) dA$$

where S is the region enclosed by the curve C. In our case this is:

$$\oint_C (e^{3x} + 2y)dx + (x^2 + \sin y)dy = \int\int_S (2x - 2)dA$$

For the square the limits of integration for the double integral are pretty easy, and this becomes:

$$\int_1^4 \int_2^6 (2x - 2)dxdy = \int_1^4 (x^2 - 2x|_2^6)dy$$
$$= \int_1^4 ((36 - 12) - (4 - 4))dy = \int_1^4 24dy = 24(4 - 1) = 72$$

6. (10 points)

For the function:

$$f(x, y) = e^{-(x^2+y^2-4y)}$$

find all the critical points, and indicate whether each such point gives a local maximum, a local minimum, or a saddle point.

Solution

Our function is a composition of differentiable elementary functions defined on the whole xy-plane, and so there is no boundary for the space in which it's defined, and there are no points at which it is not differentiable. So, the only critical points are the points where $\triangle f(x, y) = 0$.

Now, the two partial derivatives of our function are:

$$\frac{\partial f}{\partial x} = -2xe^{-(x^2+y^2-4y)}$$

and

$$\frac{\partial f}{\partial y} = (4 - 2y)e^{-(x^2+y^2-4y)}$$

Now e^x is never 0, and so there are no values (x, y) such that $e^{-(x^2+y^2-4y)} = 0$. Therefore, the only place where $\frac{\partial f}{\partial x} = 0$ is where $-2x = 0$. Similarly, the only place where $\frac{\partial f}{\partial y} = 0$ is where $(4-2y) = 0$. These points are where $x = 0$ and $y = 2$, respectively, and so the only point where both are 0 is $(0, 2)$. This is the only critical point.

Now, to check if this critical points is a maximum, minimum, or saddle point we have to calculate the second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = (4x^2 - 2)e^{-(x^2+y^2-4y)}$$

$$\frac{\partial^2 f}{\partial y^2} = (4-2y)^2 e^{-(x^2+y^2-4y)} - 2e^{-(x^2+y^2-4y)}$$

$$\frac{\partial^2 f}{\partial x \partial y} = (-2x)(4-2y)e^{-(x^2+y^2-4y)}$$

Evaluating these at $(0,2)$ we get:

$$\frac{\partial^2 f(0,2)}{\partial x^2} = -2e^4$$

$$\frac{\partial^2 f(0,2)}{\partial y^2} = -2e^4$$

$$\frac{\partial^2 f(0,2)}{\partial x \partial y} = 0$$

So, our discriminant D is:

$$D = (\frac{\partial^2 f}{\partial x^2})(\frac{\partial^2 f}{\partial y^2}) - (\frac{\partial^2 f}{\partial x \partial y})^2 = (-2)(-2) - 0^2 = 4$$

So, $D > 0$ and we know we have an extremum. As $\frac{\partial^2 f}{\partial x^2} < 0$ we know this must be a local maximum. It's actually a global maximum as well, incidentally, although you didn't have to state this for the problem's solution.

7. (10 points)

   Find the volume of the solid bounded by the cylinders $x^2 = y$ and $z^2 = y$ and the plane $y = 1$.

   Hint - Your x and z limits depend only on y.

   Solution

   We note that, for a set value of y our x coordinate is allowed to vary between $-\sqrt{y}$ and $\sqrt{y}$, and the same with our z coordinate. The x and z limits are independent of each other, and depend only on y. We also note that y must be positive, and its maximum possible value is determined by the plane $y = 1$. So, setting up the triple integral, with our y coordinate being our final variable of integration we get:

$$\int\int\int_S dV = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_{-\sqrt{y}}^{\sqrt{y}} dzdxdy$$

$$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} 2\sqrt{y}dxdy = \int_0^1 (2\sqrt{y})(2\sqrt{y})dy = \int_0^1 4ydy = 2y^2|_0^1 = 2$$

   P.S. - I'm not going to confuse you more by attempting to draw the picture. It's beyond my skill. This problem just required some reasoning, not great artistic ability.

8. (10 points)

For the vector field:

$$\mathbf{F} = (6xy^3 + 2z^2)\mathbf{i} + (9x^2y^2)\mathbf{j} + (4xz + 1)\mathbf{k}$$

defined on all of 3-space:

(a) (3 points)

Prove that the vector field is conservative by demonstrating that its curl is identically 0.

Solution

We have our vector field:

$$\mathbf{F} = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k} = \\ (6xy^3 + 2z^2)\mathbf{i} + (9x^2y^2)\mathbf{j} + (4xz + 1)\mathbf{k}$$

The curl is defined as:

$$\triangle \times \mathbf{F} = (\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z})\mathbf{i} + (\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x})\mathbf{j} + (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mathbf{k}$$

For our vector field:

$$\frac{\partial M}{\partial y} = 18xy^2 \; \frac{\partial M}{\partial z} = 4z$$
$$\frac{\partial N}{\partial x} = 18xy^2 \; \frac{\partial N}{\partial z} = 0$$
$$\frac{\partial P}{\partial x} = 4z \; \frac{\partial P}{\partial y} = 0$$

So, the curl is:

$$\triangle \times \mathbf{F} = (0 - 0)\mathbf{i} + (4z - 4z)\mathbf{j} + (18xy^2 - 18xy^2)\mathbf{k} \\ = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

So, the curl of our vector field is 0, and therefore it's conservative.

(b) (4 points)
Figure out the generating scalar function f such that $\bigtriangledown f = \mathbf{F}$.

Solution

We want a function f such that $\triangle f = \mathbf{F}$. Therefore, focusing on the **i** term, we must have:

$$\frac{\partial f}{\partial x} = 6xy^3 + 2z^2$$

Integrating this with respect to x we get:

$$\int \frac{\partial f}{\partial x} dx = \int (6xy^3 + 2z^2) dx = 3x^2y^3 + 2xz^2 + C(y,z)$$

where $C(y,z)$ is an unknown function of the variables y and z.

Now, focusing on the **j** term of our vector field and using the above inforamation we have:

$$\frac{\partial f}{\partial y} = 9x^2y^2 + \frac{\partial C(y,z)}{\partial y} = 9x^2y^2$$

from which we can conclude:

$$\frac{\partial C(y,z)}{\partial y} = 0$$

and therefore:

$$C(y,z) = C(z)$$

where $C(z)$ is an unknown function of z.

Finally, focusing on the **k** term of our vector field and using our earlier information we have:

$$\frac{\partial f}{\partial z} = 4xz + \frac{dC(z)}{dz} = 4xz + 1$$

from which we conclude:

$$\frac{dC(z)}{dz} = 1$$

and integrating both sides of this we get:

$$C(z) = \int \frac{dC(z)}{dz} dz = \int 1 dz = z + C$$

where C is an unknown constant.

Combining all this information we get:

$$f(x, y, z) = 3x^2y^3 + 2xz^2 + z + C$$

(c) (3 points)

Calculate the line integral from the point $(0, 0, 0)$ to the point $(1, 1, 1)$ along any path using any method you wish.

Solution

As the vector field is conservative, there are many ways of calcuating the line integral. In fact, any line integral that begins at $(0, 0, 0)$ and ends at $(1, 1, 1)$ will work, or you could use the fundamental theorem of line integrals. I'll demonstrate two ways you could solve this, although I'd accept as correct any line integral done correctly with the proper end points.

Solution 1

Using the fundamental theorem of line integrals we have:

$$\int_{(0,0,0)}^{(1,1,1)} \mathbf{F} \cdot d\mathbf{r} = f(1, 1, 1) - f(0, 0, 0)$$

$$= [3(1^2)(1^3)+2(1)(1^2)+(1)+C]-[3(0^2)(0^3)+2(0)(0^2)+(0)+C] = 6$$

This is probably the easiest way to do it.

Solution 2

Here I will take the line integral along the three sides of a cube shown above. We can break this up into 3 line integrals.

Along the first line segment we have $y = z = dy = dz = 0$ and so our line integral is:

$$\int_0^1 (6x(0^3) + 2(0^2)dx = \int_0^1 0dx = 0$$

Along the second line segment we have $x = 1$ and $z = dx = dz = 0$ and so our line integral is:

$$\int_0^1 (9(1^2)y^2 dy = \int_0^1 9y^2 dy = 3y^3|_0^1 = 3$$

Now, along the third line segment we have $x = y = 1$ and $dx = dy = 0$, and so our line integral is:

$$\int_0^1 (4(1)z + 1)dz = \int_0^1 (4z + 1)dz = (2z^2 + z)|_0^1 = 3$$

So, adding up our three line integrals to get the total line integral we get:

$$0 + 3 + 3 = 6$$

The two solutions agree, as we know they must. Note that other line integrals along, for example, a straight line connecting $(0, 0, 0)$ to $(1, 1, 1)$ or a parabolic path connecting the two points could have worked too and would have also necessarily given us the same result.

9. (10 points)

Evaluate the integral $\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$.

(Note - You must provide a formal evaluation of any integral. For full credit you can't just quote a result from the textbook or from lecture, you must rederive the result. Also, your final answer may be in terms of μ and σ.)

Solution

This is just problem 5 from Quiz 2. The solution is reprinted below:

If we begin with the substitution $u = \frac{(x-\mu)}{\sqrt{2}\sigma}$ then we have $du = \frac{dx}{\sqrt{2}\sigma}$ and the integral becomes:

$$\int_{-\infty}^{\infty} \sqrt{2}\sigma e^{-u^2} du = \sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-u^2} du$$

Now, this integral is tricky, but we went over it in class, in the review session, and it's covered in the textbook. So, here's how it's done. First, we take a look at the double integral:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = [\int_{-\infty}^{\infty} e^{-x^2} dx]^2$$

So, if we can calculate the double integral above, its square root will be the integral for which we're searching. The trick here is to convert the above double integral into a polar integral over the entire plane:

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r dr d\theta = \int_0^{2\pi}\int_0^{\infty} \frac{e^{-u}}{2} du d\theta$$

where here we used the substitution $u = r^2$ and so $du = 2rdr$

$$\int_0^{2\pi} -\frac{e^{-u}}{2}|_0^{\infty} du d\theta = \int_0^{2\pi} (-0 - (-\frac{1}{2}))d\theta = 2\pi(\frac{1}{2}) = \pi$$

So, the double integral above is equal to π, and therefore the single integral for which we're looking is equal to $\sqrt{\pi}$. Using this we get:

$$\sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{2\pi}\sigma$$

10. (10 points)

Prove the identity:

$$\int_0^x \int_0^v \int_0^u f(t)dtdudv = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt$$

Hint - Switch the order of integration.

Solution

This is a problem from *Calculus* by Tom Apostol, which is probably the most difficult calculus textbook ever written. So, don't feel too bad if you didn't get it right. I didn't expect many people would. I didn't get it right when I was a freshman, either. If you did get it right, give yourself a pat on the back. Anyways, it turns out to not be too hard.

We can start by looking at the inner two integrals:

$$\int_0^v \int_0^u f(t)dtdu$$

If we look at the region of integration this defines, it looks like this:

Note - This is very similar to a webworks problem we did where we had to switch the limits of integration.

If we switch our limits of integration, we get:

$$\int_0^v \int_0^u f(t)dtdu = \int_0^v \int_t^v f(t)dudt$$

and we can take the first inner integral to get:

$$\int_0^v (v-t)f(t)dt$$

Now, combining this with our entire equation we now have something that looks like this:

$$\int_0^x \int_0^v (v-t)f(t)dtdv$$

We can switch the limits of integration in exactly the same way as before to get:

$$\int_0^x \int_0^v (v-t)f(t)dtdv = \int_0^x \int_t^x (v-t)f(t)dvdt$$

Taking the inner integral we get:

$$= \int_0^x \frac{(v-t)^2}{2}\Big|_t^x f(t)dt = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt$$

Which is what we need to prove. So, we have derived the relation:

$$\int_0^x \int_0^v \int_0^u f(t)dtdudv = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt$$

Which is what we want to prove. Thanks for a great summer everybody. Good luck in all you do.

-Dylan