

Math 2280 - Lecture 7: Population Models

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Today we're going to explore one of the major applications of differential equations - population models. We'll also explore these models tomorrow in the context of autonomous differential equations.

The exercises for section 2.1 are:

Section 2.1 - 1, 8, 11, 16, 29

Population Models

Population growth models depends upon two parameters - birth and death. We call β the "birth rate" of the population, and δ the "death rate" of the population. The population model is

$$\frac{dP}{dt} = (\beta - \delta)P.$$

The terms β and δ are not necessarily constants, and could themselves be functions of time or the population size.

The simplest population model is one in which β and δ are constant. In this scenario the rate of growth of the population is directly proportional to the population. A population with this characteristic is modeled by the differential equation

$$\frac{dP}{dt} = kP.$$

There are many things in life that actually do fit this model¹, and it can be a reasonable short-term model for population growth. However, the thing about exponentials is they grow really, really fast, and no real population can keep growing exponentially forever. Something, usually the exhaustion of resources, enters the picture to correct things.

So, let's look at a slightly more sophisticated population model. Suppose the birth rate of our population is a *linear* decreasing function of the population size. So,

$$\beta = \beta_0 - \beta_1 P,$$

while the death rate is constant,

$$\delta = \delta_0.$$

In this case the population model will be

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0)P,$$

which we can rewrite as

$$\frac{dP}{dt} = aP - bP^2.$$

If the parameters a and b are both positive then the above equation is called the *logistic equation*. It's actually even more convenient to rewrite it as

$$\frac{dP}{dt} = kP(M - P)$$

where $k = b$ and $M = a/b$.

¹Compound interest, for example.

Example - Solve the logistic growth equation with initial condition $P(0) = P_0 > 0$.

Consider a population $P(t)$ of unsophisticated animals in which females rely solely on chance encounters to meet males for reproductive purposes.² Here the rate of growth will be proportional to the product of the number of males, $P/2$, and the number of females, $P/2$. So, births will occur at a rate kP^2 , and the “birth rate” will be kP . If the death rate, δ , is constant then the differential equation modeling our population will be

$$\frac{dP}{dt} = kP^2 - \delta P = kP(P - M).$$

This type of differential equation exhibits very interesting behavior.

Example - Solve the differential equation above, and explain why it’s sometimes called the “Doomsday-Extinction” model.

²For example, bars on weekends...

More room for the example problem.

Notes on Homework Problems

For problems 2.1.1 and 2.1.8 you're just asked to solve a separable differential equation like the ones you saw in section 1.4. Shouldn't be too bad.

For problem 2.1.11 please remember that the birth rate and death rate, β and δ respectively, are both multiplied by P for our population growth differential equation $\frac{dP}{dt} = (\beta - \delta)P$.

For problem 2.1.16 you may want to use a result derived in problem 2.1.15. Namely, that the limiting population is $M = \frac{B_0 P_0}{D_0}$.

Finally, in problem 2.1.29 we look at some real world data, namely census data for the United States, and examine how well our logistic growth equation models real world growth. You should find that, while the model was pretty good up to about 1940, after that things changed, and our population today is actually greater than the model would predict. So, what gives? Well, there are a lot of reasons why this might be the case, but the greatest reason is that in the last 60 years science has advanced dramatically, and so we've, in effect, been able to modify the carrying capacity of our environment. Way to go, homo sapiens!