Math 2280 - Lecture 43

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Suppose a two-dimensional object, or lamina, occupies a region R in the xy-plane. Under some reasonable assumptions we can derive that the flow of temperature through this object will satisfy the partial differential equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}\right).$$

If we assume there have been no major changes to the system for a while, the system will be in a *steady-state*, where the temperature at a given point is not changing in time. In this situation our differential equation will satisfy:

$$0 = \frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}\right).$$

The equation on the right (divided by k) is the *two-dimensional Laplace equation*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

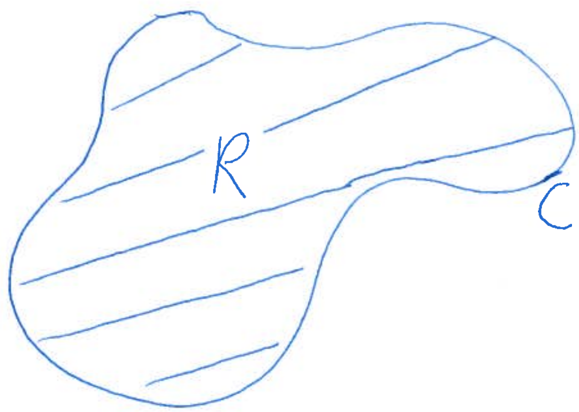
It is also known as the *potential equation*. Today, we're going to talk about how to solve this equation in some simple situations.

This lecture corresponds with section 9.7 from the textbook, and the assigned problems from this section are:

Section 9.7 - 1, 2, 3, 4

Dirichlet Problems

Suppose we have a bounded plane region R whose boundary is a continuous curve C:



A *Dirichlet problem* is a problem that asks you to solve the Laplace equation in the region R given boundary values on the curve C:

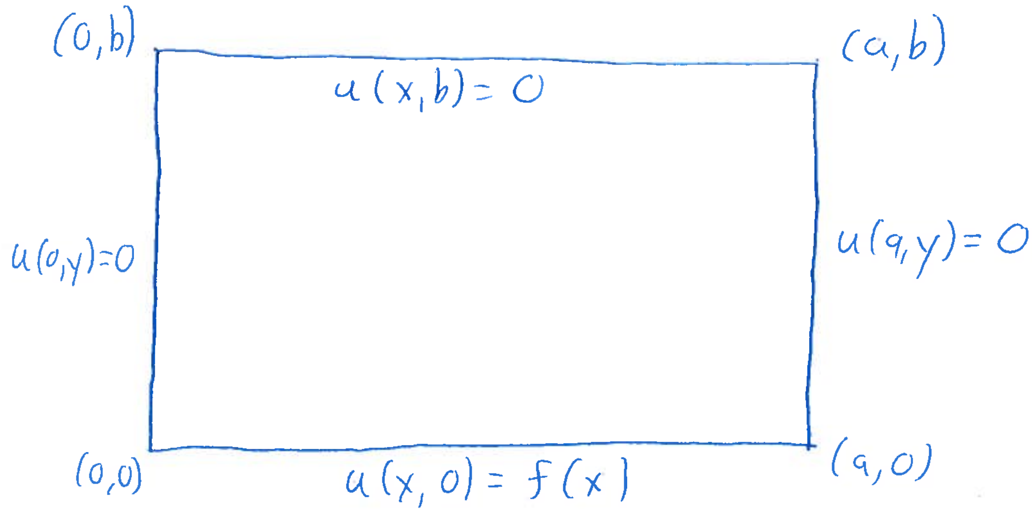
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{within } R;$$
$$u(x, y) = f(x, y) \quad \text{on } C.$$

If the boundary curve C and the boundary value function f are reasonably well behaved, then there exists a unique solutions to the Dirichlet problem. Note that "reasonably well behaved" is, of course, defined much more precisely in more advanced classes where you'd actually prove this existence and uniqueness result, but here just know that for all the problems we'll consider things are reasonably well behaved.

Dirichlet Problem in a Rectangle

Example - Solve the boundary value problem

$$u_{xx} + u_{yy} = 0;$$
$$u(0, y) = u(a, y) = u(x, b) = 0;$$
$$u(x, 0) = f(x).$$



Solution - In general for a problem of this type we'd find *four* solutions, one for each side of the rectangle, with the other sides set to 0, and then we'd add up the four solutions to get our final solution. So, this problem is not as limited as you might think.

We will, as usual, use separation of variables here. So, assume our solution is of the form

$$u(x, y) = X(x)Y(y).$$

Under this assumption our differential equation gives us:

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

for some constant λ. So, $X(x)$ must satisfy, again, the by now very familiar eigenvalue problem:

$$X'' + \lambda X = 0$$
$$X(0) = X(a) = 0.$$

The eigenvalues and associated eigenfunctions are:

$$\lambda_n = \frac{n^2\pi^2}{a^2} \quad X_n = \sin\left(\frac{n\pi x}{a}\right).$$

The differential equation for Y, along with the remaining homogeneous boundary condition, gives us:

$$Y_n'' - \frac{n^2\pi^2}{a^2} Y_n = 0 \quad Y_n(b) = 0.$$

The general solution to this differential equation, using hyperbolic sines and cosines instead of exponentials, is:

$$Y_n(y) = A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right).$$

The endpoint condition $Y_n(b) = 0$ implies that:

$$B_n = -A_n \left(\frac{\cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi y}{a}\right) \right).$$

From this, after using some trig identities, we get:

$$\begin{aligned} Y_n(y) &= A_n \cosh\left(\frac{n\pi y}{a}\right) - A_n \left(\frac{\cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi y}{a}\right) \right) \sinh\left(\frac{n\pi y}{a}\right) \\ &= \frac{A_n}{\sinh\left(\frac{n\pi b}{a}\right)} \left(\sinh\left(\frac{n\pi b}{a}\right) \cosh\left(\frac{n\pi y}{a}\right) - \cosh\left(\frac{n\pi b}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \right) \\ &= c_n \sinh\left(\frac{n\pi(b-y)}{a}\right), \end{aligned}$$

where $c_n = \dfrac{A_n}{\sinh\left(\frac{n\pi b}{a}\right)}$.

From this we get our formal series solution:

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b-y)}{a}\right).$$

It remains only to find the coefficients c_n that satisfy the nonhomogeneous condition

$$u(x,0) = \sum_{n=1}^{\infty} \left(c_n \sinh\left(\frac{n\pi b}{a}\right)\right) \sin\left(\frac{n\pi x}{a}\right) = f(x).$$

Using the orthogonality of the sine functions we get:

$$c_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx.$$

Example - Solve the boundary value problem

$$u_{xx} + u_{yy} = 0;$$
$$u(0, y) = u(a, y) = u(x, b) = 0;$$
$$u(x, 0) = x.$$

Solution - This is the example problem above, with $f(x) = x$. In this situation the coefficients c_n will be:

$$c_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a x \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \left(\frac{a^2}{n^2\pi^2} \sin\left(\frac{n\pi x}{a}\right) - \frac{ax}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\right)\Bigg|_0^a = \frac{2a(-1)^{n+1}}{n\pi \sinh\left(\frac{n\pi b}{a}\right)}.$$

The corresponding solution will be:

$$u(x, y) = \sum_{n=1}^{\infty} \left(\frac{2a(-1)^{n+1}}{n\pi \sinh\left(\frac{n\pi b}{a}\right)}\right) \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(b - y)}{a}\right).$$

Notes on Homework Problems

As mentioned, the Dirichlet problem we solved is not as restricted as you might imagine, and to solve the problem in general for a rectangular region we'd combine four solutions, one for each side. The solution for one side is done as an example problem above. The solution for the other three sides are problems 9.7.1, 9.7.2, and 9.7.3. These are all very similar to the example problem.

Problem 9.7.4 investigates a different type of boundary value problem, where instead of specifying the value of the function on all sides, on some sides the value of the function's normal derivative (the derivative in the direction away from the region) is specified. A problem where derivatives are specified like this is called a *Neumann* boundary-value problem. Problem 9.7.4 investigates one such problem.