Math 2280 - Lecture 41

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Lecture 40 is a long one, so we won't quite be able to get through it in one day. Consequently, today's lecture is a short one, so that we have time at the beginning of class to finish lecture 40.

In this lecture we'll finish up our discussion of the one-dimensional heat equation by looking at the solutions we get with some different endpoint conditions, these ones involving derivatives.

Insulated Endpoint Conditions

Suppose instead of the endpoint conditions $u(0, t) = u(L, t) = 0$ we have the conditions $u_x(0, t) = u_x(L, t) = 0$. So, no heat flows into or out of our rod. These are still homogeneous boundary conditions, so an approach similar to the one we used last lecture should work.

Formally, we want to find a solution to the boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, (0 < x < L, t > 0);$$

$$u_x(0, t) = u_x(L, t) = 0,$$

$$u(x, 0) = f(x).$$

The way we solve this is exactly the same way we did before (separation of variables) except with the different endpoint conditions. What we end up with is that our eigenvalues are going to be:

$$\lambda_n = \frac{n^2 \pi^2}{L^2},$$

and the associated eigenfunctions will be:

$$X_n = \cos \frac{n \pi x}{L}.$$

Just as before the solution for our time function will be:

$$T_n(t) = e^{-\frac{n^2 \pi^2 kt}{L^2}},$$

and our solutions will be:

$$u_n(x, t) = e^{-\frac{n^2 \pi^2 kt}{L^2}} \cos \frac{n \pi x}{L}.$$

So, the total solution will be:

$$u(x, t) = \sum_{n=0}^{\infty} c_n e^{\frac{-n^2 \pi^2 kt}{L^2}} \cos \frac{n \pi x}{L}.$$

To satisfy the initial condition $u(x, 0) = f(x)$ we want this to reduced, when $t = 0$, to the Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n \pi x}{L},$$

where the Fourier coefficients are given by

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n \pi x}{L} dx.$$

Example - Suppose we have a 50-cm rod, whose initial temperature is given by the "triangular function" :

$$f(x) = \begin{cases} 4x & 0 \leq x \leq 25 \\ 200 - 4x & 25 \leq x \leq 50 \end{cases}$$

and its temperature satisfies the boundary value problem:

$$u_t = ku_{xx},$$

$$u_x(0, t) = u_x(50, t) = 0,$$

$$u(x, 0) = f(x).$$

Find the solution to this boundary value problem.

Solution - The Fourier cosine series for the initial temperature will be

$$f(x) = 50 - \frac{400}{\pi^2} \sum_{n \ odd} \frac{1}{n^2} \cos \frac{n\pi x}{25}.$$

In order for it to match the terms in the series given above with $L = 50$ we must have terms of the form $\cos(n\pi x/50)$. So, we replace n with $n/2$ throughout and rewrite the series in the form

$$f(x) = 50 - \frac{1600}{\pi^2} \sum_{n=2,6,10,...} \frac{1}{n^2} \cos \frac{n\pi x}{50}.$$

So, our solution will be:

$$u(x, t) = 50 - \frac{1600}{\pi^2} \sum_{n=2,6,10,...} \frac{1}{n^2} e^{-\frac{n^2\pi^2 kt}{2500}} \cos \frac{n\pi x}{50}.$$

Note that as $t \to \infty$ all the terms in the sum go to 0, and the temperature of the rod approaches a uniform 50 degrees.

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