Math 2280 - Lecture 35

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In the last lecture we learned how to solve linear ODEs of the form:

$$A(x)y'' + B(x)y' + C(x)y = 0,$$

around regular singular points using Frobenius series. The case we put off for later was when the two roots of the indicial equation differ by an integer. Today, we'll learn what to do in that situation.

Today's lecture corresponds with section 8.4 of the textbook. The assigned problems are:

Section 8.4 - 1, 6, 8, 9, 14

Method of Frobenius : The Exceptional Cases

The difference $r_1 - r_2$, with $r_1 \geq r_2$, is an integer if $r_1 = r_2$, or if $r_1 = r_2 + N$. In the latter case there might, or might not, be two Frobenius solutions. In the former case there's obviously only one Frobenius solution.

First, let's work an example where they differ by an integer and we get two distinct Frobenius solutions.

Example - Solve the ODE:

$$x^2 y'' + (6x + x^2)y' + xy = 0.$$

More room for the example problem.

So, everything works great, right? Unfortunately not. Here's a situation where this method fails.

Example - Does the Frobenius method provide us two linearly independent solutions to the linear ODE:

$$x^2y'' - xy' + (x^2 - 8)y = 0?$$

More room for the example problem.

But, how do we find the other solution?

Reduction of Order

Suppose we have an ODE

$$y'' + P(x)y' + Q(x)y = 0$$

and we've derived one solution, $y_1(x)$. We want to find another solution, $y_2(x)$.

First we note that finding $y_2(x)$ is equivalent to finding:

$$v(x) = \frac{y_2(x)}{y_1(x)}.$$

Here we're assuming $y_1(x) \neq 0$ on our interval of interest. If $y_2(x) = y_1(x)v(x)$ then:

$$y_2'(x) = y_1(x)v'(x) + y_1'(x)v(x)$$

and

$$y_2''(x) = v(x)y_1''(x) + 2y_1'(x)v'(x) + y_1(x)v''(x).$$

If we plug this into our ODE and do some grouping we get:

$$[v(x)y_1''(x) + 2v'(x)y_1'(x) + v''(x)y_1(x)]$$
$$+P(x)[y_1(x)v'(x) + y_1'(x)v(x)] + Q(x)v(x)y_1(x) = 0,$$
$$\Rightarrow v(x)(y_1''(x) + P(x)y_1'(x) + Q(x)y_1(x))$$
$$+v''(x)y_1(x) + 2v'(x)y_1'(x) + P(x)v'(x)y_1(x) = 0.$$

The first term above is, by definition, 0. This gives us the relation:

$$v''(x)y_1(x) + (2y_1'(x) + P(x)y_1(x))v'(x) = 0.$$

If we write $u(x) = v'(x)$ then we get an equation we know how to solve. Namely,

$$u'(x) + \left(2\frac{y_1'(x)}{y_1(x)} + P(x)\right)u(x) = 0.$$

This is a first-order linear ODE. To solve it, we introduce the integrating factor:

$$\rho = e^{\int (2\frac{y_1'(x)}{y_1(x)} + P(x))dx} = e^{2\ln|y_1(x)| + \int P(x)dx} = (y_1(x))^2 e^{\int P(x)dx}.$$

Integrating our ODE gives us:

$$u(x)(y_1(x))^2 e^{\int P(x)dx} = C.$$

Noting $u(x) = v'(x)$ we integrate for $v(x)$ to get:

$$v(x) = C\int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx + K.$$

If we use $C = 1$ and $K = 0$ then we finally get our solution:

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}dx.$$

So, with one solution, we can find a second.

Example - Find a second solution to the ODE:

$$x^2 y'' - 9xy' + 25y = 0$$

given that x^5 is one solution.

Now, how do we apply this method to our special Frobenius cases. Well, the textbook goes over a very long derivation that combines power series and the reduction of order method to figure out the form of the second solution. You can read it there. The punchline is the following theorem.

Theorem - If you have the ODE:

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y = 0$$

where $p(x)$ and $q(x)$ are analytic around $x = 0$, and the roots of the indicial equation, r_1, r_2 with $r_1 \geq r_2$, differ by an integer then one solution is:

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} c_n x^n.$$

As for the other solution, if $r_1 = r_2$, then:

$$y_2(x) = y_1(x) \ln x + x^{r_1+1} \sum_{n=0}^{\infty} b_n x^n.$$

On the other hand, if $r_1 \neq r_2$, then:

$$y_2(x) = C y_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n.$$

Here C *may* or *may not* (we've seen examples of both) be 0.

The radius of convergence for our general solution is at least ρ, the minimum of the radii of convergence for $p(x)$ and $q(x)$.

Applying this theorem boils down to the same techniques we've seen before. Plug it in, find the recursion relations for the coefficients, and then see if you can get them into a closed form.

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Notes on Homework Problems

Dang! This section gets complicated.

Problems 8.4.1, 8.4.6, and 8.4.8 are like the example problems from these notes. You either need to find two linearly independent Frobenius series solutions, or find one and prove the other one doesn't exist. Note that you do not have to use reduction of order to find a second linearly independent solution if a second Frobenius series solution does not exist.

On problems 8.4.9 and 8.4.14 you don't need to find a complete Frobenius series solution, but you will need to find the first four terms. Then, you'll use these to initial terms of a second solution found using the reduction or order technique. These problems... require some time, some concentration, some fastidiousness, some algebra, and some paper. Get to it!