Math 2280 - Lecture 11

Dylan Zwick

Spring 2014

Up to this point we've focused almost exclusively on first-order differential equations, with only passing references to differential equations of higher order. Starting today, this will change. In this lecture we'll have our first substantial discussion of second-order differential equations. We'll discuss the necessary and important existence and uniqueness theorem, and then learn how to solve these differential equations in some simple, but still very useful, situations.

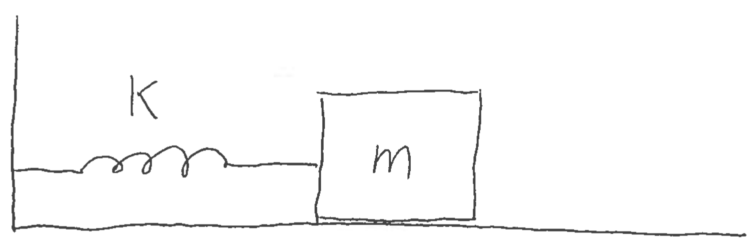
The assigned problems for this section are:

Section 3.1 - 1, 16, 18, 24, 39

Second-Order Linear Equations

Initial Example

Suppose we take a mass m and attach it to a spring:



If we displace the mass a short distance x from its equilibrium there will be a restorative force F acting on it, $F = -kx$, where k is the "spring constant". This is called Hooke's law. If we combine Hooke's law with Newton's second law[1] we get:

$$F = -kx = m\frac{d^2x}{dt^2}.$$

So, we have the relation:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

One solution to this second-order ODE is:

$$x(t) = \sin\left(\sqrt{\frac{k}{m}}t\right),$$

and another is

$$x(t) = \cos\left(\sqrt{\frac{k}{m}}t\right).$$

In fact, any linear combination

$$x(t) = c_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + c_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

works as a solution. This raises some questions:

1. Does this cover *all* solutions to this ODE?
2. Does this handle *all* possible initial conditions for displacement and velocity?
3. Does this situation generalize?

Yes is the answer to all three questions.

[1]Isaac Newton and Robert Hooke were, incidentally, contemporaries. And they hated each other. But, that's hardly surprising, as Newton hated almost everybody.

General Theory of Second-Order Differential Equations

We call a differential equation of the form:

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

a linear second-order ODE. Note that the functions A, B, C, and F aren't necessarily linear.

We'll usually be interested in finding a solution on an (possibly unbounded) interval I. If $F(x) = 0$ on I then we call the second-order linear ODE *homogeneous*.

Our initial example was indeed a homogeneous second-order ODE with:

$$A(x) = m$$
$$B(x) = 0$$
$$C(x) = k$$

Now, we saw that we can find two different functions that both solved the ODE, and in fact any linear combination of these functions also solved the ODE. This is true in general.

Theorem - For any homogenenous linear second-order ODE with solutions y_1, y_2 on I any function of the form

$$y = c_1y_1 + c_2y_2$$

is also a solution on I.

This theorem is pretty obvious and can be checked quite easily. It follows almost immediately from the linearity of the derivative.

$$A(x)(c_1y_1 + c_2y_2)'' + B(x)(c_1y_1 + c_2y_2)' + C(x)(c_1y_1 + c_2y_2)$$
$$= c_1(A(x)y_1'' + B(x)y_1' + C(x)y_1) + c_2(A(x)y_2'' + B(x)y_2' + C(x)y_2)$$
$$= c_1(0) + c_2(0) = 0.$$

As for questions of existence and uniqueness, just as with linear first-order ODEs we have an existence and uniqueness theorem:

Theorem - Suppose that functions p, q and f are continuous on the (possibly unbounded) open interval I containing the point a. Then, given any two numbers b_0, b_1 the equation:

$$y'' + p(x)y' + q(x)y = f(x)$$

has a unique solution on all of I that satisfies:

$$y(a) = b_0,\ y'(a) = b_1.$$

Example - Verify that the two given solutions are in fact solutions to the second-order differential equation given below, and then find a linear combination of these two solutions such that the initial conditions are satisfied.

$$y'' - 9y = 0;$$
$$y_1 = e^{3x}, y_2 = e^{-3x}; y(0) = -1, y'(0) = 15.$$

Solution - If we plug in the two solutions we get:

$$y_1'' - 9y_1 = 9e^{3x} - 9e^{3x} = 0.$$

So, y_1 checks out. Doing the same with y_2 we get:

$$y_2'' - 9y_2 = 9e^{-3x} - 9e^{-3x} = 0.$$

So, y_2 checks out as well. A linear combination of the two solutions will be of the form:

$$y = c_1y_1 + c_2y_2 = c_1e^{3x} + c_2e^{-3x},$$
$$y' = c_1y_1' + c_2y_2' = 3c_1e^{3x} - 3c_2e^{-3x}.$$

If we plug in $x = 0$ we get:

$$y(0) = c_1 + c_2,$$

and

$$y'(0) = 3c_2 - 3c_2.$$

So, our initial conditions give us the two linear equations:

$$c_1 + c_2 = -1$$

$$3c_1 - 3c_2 = 15.$$

Solving for c_1 and c_2 we get $c_1 = 2$, $c_2 = -3$. So, the solution to the initial value problem is:

$$y(x) = 2e^{3x} - 3e^{-3x}.$$

Linear Independence of Two Functions

First, a definition.

Definition - Two functions f, g defined on an open interval I are linearly independent on I provided that neither is a constant multiple of the other.

A pair of functions are linearly dependent if they're not linearly independent.[2]

For two functions f and g we define a third function called the *Wronskian*:

$$W(x) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'.$$

Why do we do this? Here's why. If f and g are linearly dependent then

$$W(f,g) = 0 \text{ on } I.$$

On the other hand, if f and g are two linearly independent solutions to a second-order ODE then

$$W(f,g) \neq 0 \text{ on } \textit{every} \text{ point of } I.$$

That *every* point bit is the important and amazing part.

We'll end by addressing a question about whether or not we've found all the solutions to a given linear second-order ODE. If y_1 and y_2 are linearly independent solutions to a linear second-order ODE then *all* solutions to the ODE are of the form:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

This can be proven without too much problem by using our existence and uniqueness theorem along with some linear algebra. It's done in the textbook.

[2] Well... duh!

Second-Order Linear Homogeneous ODEs with Constant Coefficients

A linear homogeneous second-order ODE is an ODE of the form:

$$ay'' + by' + cy = 0$$

where a, b, c are constant.

If we try the solution $y(x) = e^{rx}$ and plug it in we get:

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

Dividing through by e^{rx} we see that this solution works if r is a root of the quadratic equation:

$$ax^2 + bx + c = 0.$$

So,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We'll only deal with distinct real roots for now. If the roots are distinct real numbers, then all our solutions are of the form:

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$$

where the roots are r_1 and r_2.

Example - What are all the solutions to the differential equation:

$$y''(x) + 2y'(x) - 15y(x) = 0$$

Solution - If we plug in $y = e^{rx}$ we get:

$$r^2 e^{rx} + 2re^{rx} - 15e^{rx} = 0.$$
$$\Rightarrow (r^2 + 2r - 15)e^{rx} = 0.$$

This is only possible if $r^2 + 2r - 15 = 0$. Factoring $r^2 + 2r - 15$ we get $(r + 5)(r - 3)$, and so the roots are $r = 3$ and $r = -5$.

So, the general solution to the differential equation is:

$$y(x) = c_1 e^{3x} + c_2 e^{-5x}.$$

Notes on Homework Problems

Problems 3.1.1 and 3.1.16 are straightforward. You just plug in the functions and check that they work, and then do some simple linear algebra.

Problem 3.1.18 demonstrates that superposition is not always true for solutions to an ODE. The assumption that we're dealing with linear ODEs is very important in the principle of superposition!

Problem 3.1.24 is a little bit tricky. Think trig identities!

In problem 3.1.30 demonstrates why the assumptions behind the existence and uniqueness theorem are necessary.