Math 2280 - Final Exam

University of Utah

Spring 2014

Name: Solutions by Dylan Zwick

This is a two hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

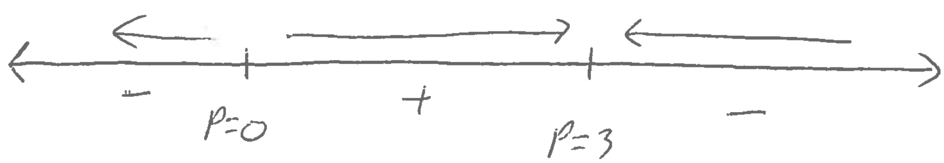
1. *Population Models*

(a) (3 points) For the population model differential equation

$$\frac{dP}{dt} = 2P(3 - P),$$

find all equilibrium values of P, draw the corresponding phase diagram, and state whether each equilibrium is stable or unstable.

Solution - The equilibrium values of P will be the values of P for which $\frac{dP}{dt} = 0$. These are the values $P = 0$ and $P = 3$. The corresponding phase diagram is



The value $P = 0$ is an unstable equilibrium, while $P = 3$ is a stable equilibrium.

(b) (7 points) Find the solution to the differential equation

$$\frac{dP}{dt} = 2P(3 - P),$$

with the initial population $P(0) = 2$. What is the limit

$$\lim_{t \to \infty} P(t)?$$

Is that what was to be expected from the phase diagram?

Solution - This is a separable differential equation, which we can rewrite as

$$\frac{dP}{P(3 - P)} = 2dt.$$

A partial fraction decomposition of the rational function on the left gives us

$$\frac{dP}{P(3 - P)} = \frac{1}{3}\left(\frac{1}{P} + \frac{1}{3 - P}\right) dP = 2dt.$$

Taking the antiderivative of both sides we get

$$\frac{1}{3}\left(\ln P - \ln\left(3 - P\right)\right) = 2t + C$$

$$\Rightarrow \ln\left(\frac{P}{3 - P}\right) = 6t + C$$

$$\Rightarrow \frac{P}{3 - P} = Ce^{6t}.$$

Solving this for $P(t)$ we get

$$P(t) = \frac{3Ce^{6t}}{1 + Ce^{6t}}.$$

From the initial condition $P(0) = 2$ we get

$$2 = P(0) = \frac{3C}{1+C}$$
$$\Rightarrow C = 2.$$

So, our solution is

$$P(t) = \frac{6e^{6t}}{1+2e^{6t}} = \frac{6}{2+e^{-6t}}.$$

We have $\lim_{t\to\infty} P(t) = 3$, which is what we'd expect from our phase diagram.

2. *Higher-Order Linear ODEs*

(a) (1 point) For the differential equation

$$y^{(3)} + 2y'' - y' - 2y = -2e^{-x} + 4x,$$

what is the order of the differential equation?

Solution - 3rd order.

(b) (1 point) Is the differential equation linear or nonlinear?

Solution - Linear.

(c) (2 points) Is the differential equation homogeneous? If not, what is the corresponding homogeneous differential equation?

Solution - It is not homogeneous. The corresponding homogeneous differential equation is

$$y^{(3)} + 2y'' - y' - 2y = 0.$$

(d) (5 points) What is the solution to the corresponding homogeneous differential equation? (*Hint*: e^x is a solution.)

Solution - The characteristic equation is $r^3 + 2r^2 - r - 2 = (r-1)(r+1)(r+2)$. So, the roots are $r = 1, -1, -2$. The corresponding homogeneous solution is

$$y_h = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}.$$

(e) (1 points) What is the form of the particular solution to the differential equation

$$y^{(3)} + 2y'' - y' - 2y = -2e^{-x} + 4x?$$

Solution - $y_p = Axe^{-x} + Bx + C$.

(f) (5 points) What is the general solution to the above differential equation?

Solution - The derivatives of y_p are

$$y_p' = -Axe^{-x} + Ae^{-x} + B,$$
$$y_p'' = Axe^{-x} - 2Ae^{-x},$$
$$y_p^{(3)} = -Axe^{-x} + 3Ae^{-x}.$$

If we plug these into the differential equation we get

$$-2Ae^{-x} - 2Bx - (B + 2C) = -2e^{-x} + 4x.$$

From these we get $A = 1, B = -2, C = 1$, and our general solution is

$$y(x) = y_h + y_p = c_1e^x + c_2e^{-x} + c_3e^{-2x} + xe^{-x} - 2x + 1.$$

3. *Laplace Transforms and Delta Functions*

(15 points) Solve the differential equation

$$x'' + 2x' + x = 1 + \delta(t),$$

with the initial conditions $x(0) = 0$, $x'(0) = 1$.

Solution - Using the derivative rule for the Laplace transform we have

$$\mathcal{L}(x(t)) = X(s),$$

$$\mathcal{L}(x'(t)) = sX(s) - x(0) = sX(s),$$

$$\mathcal{L}(x''(t)) = s^2X(s) - sx(0) - x'(0) = s^2X(s) - 1.$$

Taking the Laplace transform of both sides we get

$$(s^2 + 2s + 1)X(s) - 1 = \frac{1}{s} + 1,$$

$$X(s) = \frac{1}{s(s+1)^2} + \frac{1}{(s+1)^2}.$$

Taking a partial fraction decomposition we get

$$X(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{1}{(s+1)^2},$$

from which we get

$$x(t) = 1 - e^{-t} + te^{-t}.$$

4. *Systems of Differential Equations*

(15 points) Solve the system of differential equations

$$\begin{array}{rcrcrcr} x_1' & = & -x_1 & & & + & x_3 \\ x_2' & = & & & x_2 & - & 4x_3 \\ x_3' & = & & & x_2 & - & 3x_3 \end{array}.$$

Hint : $\lambda = -1$.

Solution - This system is equivalent to the vector equation

$$\mathbf{x}' = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{pmatrix} \mathbf{x}.$$

The eigenvalues of the coefficient matrix are the roots of the characteristic polynomial

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -4 \\ 0 & 1 & -3-\lambda \end{vmatrix} = -(\lambda+1)^3.$$

So, $\lambda = -1$ is the only eigenvalue. The corresponding eigenvector is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is, up to scaling, the only eigenvector. So, there must be a length 3 chain of generalized eigenvectors. If A is the coefficient matrix we have

$$(A-\lambda I)^2 = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(A-\lambda I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, almost any vector (as long as it's not a length 1 or 2 generalized eigenvector) will do for $\mathbf{v}_3$. If we take

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we get

$$\mathbf{v}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ -2 \end{pmatrix},$$

$$\mathbf{v}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}.$$

So, the general solution to our differential equation is

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{-t},$$

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{-t},$$

$$\mathbf{x}_3(t) = \left(\frac{1}{2}\mathbf{v}_1 t^2 + \mathbf{v}_2 t + \mathbf{v}_3\right) e^{-t},$$

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t).$$

5. *Power Series Solutions*

(15 points) Is $x = 0$ an ordinary point, a regular singular point, or an irregular singular point for the differential equation

$$y'' + x^2 y' + 2xy = 0?$$

How many linearly independent Frobenius series solutions are we guaranteed around $x = 0$? Find the general solution to the differential equation.

Solution - The functions x^2 and $2x$ are both analytic everywhere, so $x = 0$ is an ordinary point, and we're guaranteed two linearly independent Frobenius series (power series, actually) solutions that converge everywhere.

If we write our solution $y(x)$ as a power series we have

$$y(x) = \sum_{n=0}^{\infty} c_n x^n,$$

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1},$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Plugging these into our differential equation we have

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n+1} + \sum_{n=0}^{\infty} 2 c_n x^{n+1} = 0.$$

We can rewrite this equation as

$$2c_2 + \sum_{n=0}^{\infty}((n+3)(n+2)c_{n+3} + (n+2)c_n)x^{n+1} = 0.$$

From this we get

$$c_2 = 0,$$

with c_0, c_1 arbitrary, and

$$c_{n+3} = -\frac{c_n}{n+3}.$$

Using this recurrence relation we can derive

$$c_{3n} = \frac{(-1)^n c_0}{3^n n!},$$

$$c_{3n+1} = \frac{(-1)^n c_1}{1 \times 4 \times \cdots \times (3n+1)},$$

$$c_{3n+2} = 0.$$

So, our solution is

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{3^n n!} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{1 \times 4 \times \cdots \times (3n+1)}.$$

6. *Endpoint Value Problem*

(10 points) Find the eigenvalues and eigenfunctions corresponding to the non-trivial solutions of the endpoint value problem

$$X''(x) + \lambda X(x) = 0,$$

$$X'(0) = X'(3) = 0.$$

Solution - If $\lambda < 0$ then the solution to the differential equation will have the form

$$X(x) = Ae^{\sqrt{-\lambda x}} + Be^{-\sqrt{-\lambda x}},$$

$$X'(x) = A\sqrt{-\lambda}e^{\sqrt{-\lambda x}} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda x}}.$$

The endpoint condition $X'(0) = \sqrt{-\lambda}(A - B) = 0$ implies $A = B$, while the endpoint condition $X'(3) = 0$ implies

$$X(3) = A\sqrt{-\lambda}(e^{3\sqrt{-\lambda}} - e^{-3\sqrt{-\lambda}}) = 0.$$

If $A \neq 0$ this could only be true if $e^{3\sqrt{-\lambda}} = e^{-3\sqrt{-\lambda}}$, which is impossible if $\lambda < 0$. So, $A = 0$ is the only possibility, and we therefore have only the trivial solution. Thus, there are no negative eigenvalues.

If $\lambda = 0$ then the solution to our differential equation is

$$X(x) = Ax + B,$$

$$X'(x) = A.$$

The endpoint conditions $X'(0) = X'(3) = 0$ are satisfied for $A = 0$. So, B can be arbitrary, and $\lambda = 0$ is an eigenvalue with eigenfunction 1.

For $\lambda > 0$ the solution to our differential equation is

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x),$$

$$X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x).$$

For the endpoint condition $X'(0) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}0)+B\sqrt{\lambda}\cos(\sqrt{\lambda}0) = B = 0$ we get $X(x)$ must have the form

$$X(x) = A\cos(\sqrt{\lambda}x),$$

$$X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x).$$

The second endpoint condition $X'(3) = -A\sqrt{\lambda}\sin(3\sqrt{\lambda}) = 0$ tells us that, if $A \neq 0$, we must have $3\sqrt{\lambda} = n\pi$, where n is an integer. From this we get the eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{9},$$

with corresponding eigenfunctions

$$X_n(x) = \cos\left(\frac{n\pi x}{3}\right).$$

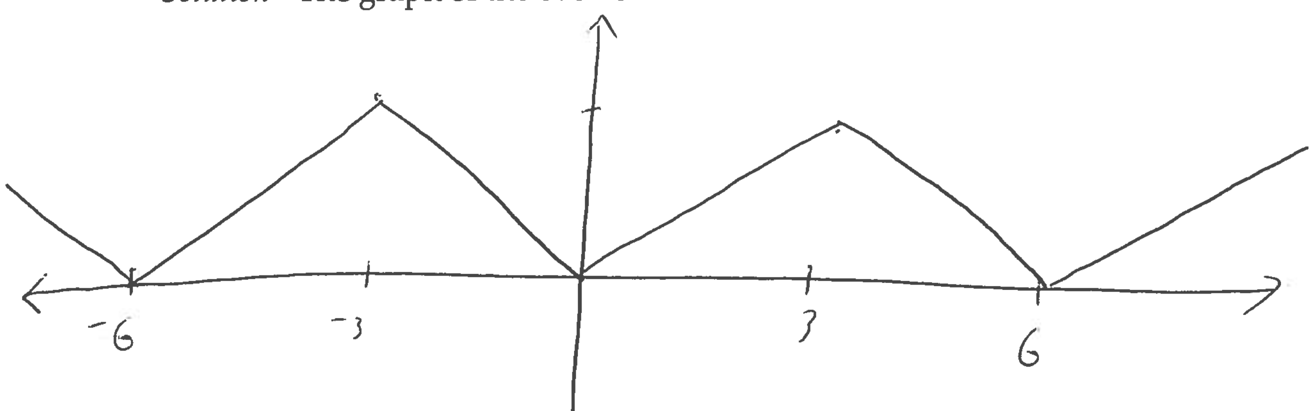
7. *Fourier Series*

(10 points) Graph the even extension of the function

$$f(x) = x \quad 0 < x < 3.$$

and find its Fourier cosine series.

Solution - The graph of the even extension looks like this:



The corresponding Fourier cosine series will have terms

$$a_0 = \frac{2}{3}\int_0^3 x dx = \left.\frac{x^2}{3}\right|_0^3 = 3 - 0 = 3,$$

and

$$a_n = \frac{2}{3}\int_0^3 x\cos\left(\frac{n\pi x}{3}\right) = \frac{2}{3}\left(\frac{3x}{n\pi}\sin\left(\frac{n\pi x}{3}\right) + \frac{9}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)\right)\Bigg|_0^3$$

$$= \frac{6}{n^2\pi^2}\left(\cos(n\pi) - 1\right) = \begin{cases} 0 & n \text{ even} \\ -\frac{12}{n^2\pi^2} & n \text{ odd} \end{cases}.$$

So, the cosine series will be

$$f(x) \sim \frac{3}{2} - \frac{12}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\left(\frac{(2n+1)\pi x}{3}\right).$$

Extra Credit - If we plug $f(0) = 0$ into this series we get, after a little algebra,

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

8. *The Heat Equation*

(10 points) Find the solution to the partial differential equation

$$u_t = 2u_{xx},$$

$$u_x(0,t) = u_x(3,t) = 0,$$

$$u(x,0) = x$$

Solution - If we assume that our equation is separable we get

$$u(x,t) = X(x)T(t).$$

In order for this to satisfy our differential equation we must have

$$X(x)T'(t) = 2X''(x)T(t)$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{2T(t)} = -\lambda,$$

where λ is some constant. So, $X(x)$ must satisfy the differential equation

$$X''(x) + \lambda X(x) = 0,$$

with our endpoint conditions $u_x(0,t) = u_x(3,t) = 0$ giving us $X'(0) = X'(3) = 0$. This is *exactly* the endpoint value problem[1] from Problem 6, and so its solution is that the eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{9},$$

[1]What a coincidence!

with eigenfunctions

$$X_n(x) = \cos\left(\frac{n\pi x}{3}\right).$$

The function $T(t)$ must satisfy the differential equation

$$T_n'(t) + 2\lambda_n T_n(t) = 0,$$

from which we get $T_n(t) = e^{-\frac{2n^2\pi^2}{9}t}$. So, our general solution will be some infinite sum

$$u(x,t) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{3}\right) e^{-\frac{2n^2\pi^2}{9}t}.$$

If we plug in our final endpoint condition we see that the c_n terms are just[2] the coefficients from Problem 7, and so our final solution is

$$u(x,t) = \frac{3}{2} - \frac{12}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi x}{3}\right) e^{-\frac{2(2n+1)^2\pi^2}{9}t}.$$

Note that as $t \to \infty$ the temperature distribution approaches the constant distrubution $\frac{3}{2}$, which is as we'd expect.

Thank you all for a great class, and good luck in all you do! - Dylan

[2] Another coincidence!