Math 2280 - Final Exam

University of Utah

Spring 2014

Name: ____

This is a two hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right).$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

1. *Population Models*

   (a) (3 points) For the population model differential equation

$$\frac{dP}{dt} = 2P(3 - P),$$

   find all equilibrium values of P, draw the corresponding phase diagram, and state whether each equilibrium is stable or unstable.

(b) (7 points) Find the solution to the differential equation

$$\frac{dP}{dt} = 2P(3 - P),$$

with the initial population $P(0) = 2$. What is the limit

$$\lim_{t\to\infty} P(t)?$$

Is that what was to be expected from the phase diagram?

2. *Higher-Order Linear ODEs*

(a) (1 point) For the differential equation

$$y^{(3)} + 2y'' - y' - 2y = -2e^{-x} + 4x,$$

what is the order of the differential equation?

(b) (1 point) Is the differential equation linear or nonlinear?

(c) (2 points) Is the differential equation homogeneous? If not, what is the corresponding homogeneous differential equation?

(d) (5 points) What is the solution to the corresponding homogeneous differential equation? (*Hint*: e^x is a solution.)

(e) (1 points) What is the form of the particular solution to the differential equation

$$y^{(3)} + 2y'' - y' - 2y = -2e^{-x} + 4x?$$

(f) (5 points) What is the general solution to the above differential equation?

3. *Laplace Transforms and Delta Functions*

(15 points) Solve the differential equation

$$x'' + 2x' + x = 1 + \delta(t),$$

with the initial conditions $x(0) = 0$, $x'(0) = 1$.

4. *Systems of Differential Equations*

(15 points) Solve the system of differential equations

$$\begin{array}{rcrcrcr} x_1' & = & -x_1 & & & + & x_3 \\ x_2' & = & & & x_2 & - & 4x_3 \\ x_3' & = & & & x_2 & - & 3x_3 \end{array} .$$

Hint : $\lambda = -1$.

More room for Problem 4, if you need it.

5. *Power Series Solutions*

(15 points) Is $x = 0$ an ordinary point, a regular singular point, or an irregular singular point for the differential equation

$$y'' + x^2y' + 2xy = 0?$$

How many linearly independent Frobenius series solutions are we guaranteed around $x = 0$? Find the general solution to the differential equation.

More room for Problem 5, if you need it.

6. *Endpoint Value Problem*

(10 points) Find the eigenvalues and eigenfunctions corresponding to the non-trivial solutions of the endpoint value problem

$$X''(x) + \lambda X(x) = 0,$$

$$X'(0) = X'(3) = 0.$$

More room for problem 6, if you need it.

7. *Fourier Series*

(10 points) Graph the even extension of the function

$$f(x) = x \quad 0 < x < 3.$$

and find its Fourier cosine series.

More room for problem 7, if you need it.

8. *The Heat Equation*

(10 points) Find the solution to the partial differential equation

$$u_t = 2u_{xx},$$

$$u_x(0,t) = u_x(3,t) = 0,$$

$$u(x,0) = x \quad 0 < x < 3.$$

More room for problem 8, if you need it.