Math 2280 - Exam 3

University of Utah

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This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

1. (10 Points) *Calculating a Laplace Transform*

   Calculate the Laplace transform of the function

   $$f(t) = t^2$$

   using the definition of the Laplace transform, and state the domain of the transform.

   Solution - Using the definition of the Laplace transform and integration by parts, we have

   $$\mathcal{L}(t^2) = \int_0^\infty e^{-st}t^2 dt = -\frac{t^2 e^{-st}}{s}\Big|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st} t dt.$$

   If $s > 0$ then the first term in the sum evaluates to 0, while if $s \leq 0$ the integral diverges. So, we'll assume $s > 0$. Starting from where we left off, and integrating by parts once again we have

   $$\mathcal{L}(t^2) = -\frac{2te^{-st}}{s^2}\Big|_0^\infty + \frac{2}{s^2}\int_0^\infty e^{-st} dt = -\frac{2e^{-st}}{s^3}\Big|_0^\infty = \frac{2}{s^3}.$$

   The domain of $F(s)$ is $s > 0$.

2. (10 Points) *Inverse Laplace Transforms*

Calculate the inverse Laplace transform of the rational function

$$F(s) = \frac{s+5}{s^2 - 2s - 3}.$$

Solution - The denominator of $F(s)$ factors as $(s-3)(s+1)$, and the partial fraction decomposition of $F(s)$ is

$$F(s) = \frac{A}{s-3} + \frac{B}{s+1} = \frac{(A+B)s + (A-3B)}{s^2 - 2s - 3}.$$

Solving for A and B here we get $A = 2$ and $B = -1$. So,

$$F(s) = \frac{2}{s-3} - \frac{1}{s+1},$$

and the inverse Laplace transform is

$$f(t) = 2e^{3t} - e^{-t}.$$

3. (15 Points) *Convolutions*

Calculate the the convolution $f(t) * g(t)$ of the following functions:

$$f(t) = t, \quad g(t) = t^2.$$

Solution - Using the definition of the convolution we have

$$f(t) * g(t) = \int_0^t \tau^2 (t - \tau) d\tau = t\frac{\tau^3}{3} - \frac{\tau^4}{4}\bigg|_0^t = \frac{t^4}{12}.$$

4. (15 Points) *Singular Points*

Determine if the point $x = 0$ is an ordinary point, a regular singular point, or an irregular singular point for the differential equation

$$x^2y'' - xy' - 3\cos(x)y = 0.$$

If $x = 0$ is a regular singular point find the roots of the indicial equation, and state whether we're guaranteed two, or just one, linearly independent Frobenius series solution.

Solution - If we divide both sides of the differential equation by x^2 we can rewrite it as

$$y'' - \frac{1}{x}y' - \frac{3\cos(x)}{x^2}y = 0.$$

The functions $P(x) = -\dfrac{1}{x}$ and $Q(x) = -\dfrac{3\cos(x)}{x^2}$ are both singular at $x = 0$, and therefore $x = 0$ is a singular point.

The functions $p(x) = xP(x) = -1$ and $q(x) = x^2Q(x) = -3\cos(x)$ are both analytic at $x = 0$, and therefore $x = 0$ is a *regular singular point*.

We have $p_0 = p(0) = -1$, and $q_0 = q(0) = -3$, and so the indicial equation is

$$r(r-1) + p_0r + q_0 = r(r-1) - r - 3 = r^2 - 2r - 3 = (r+1)(r-3).$$

So, the roots of the indicial equation are $r = -1, 3$. As these differ by an integer, we're guaranteed one Frobenius series solution, for $r = 3$, which will actually be a power series solution. We're not, however, guaranteed a second linearly independent Frobenius series solution for $r = -1$, although we might get one.

5. (20 Points) *Differential Equations and Laplace Transforms*

   Find the solution to the initial value problem

$$x'' - 4x' + 4x = \delta(t) + \delta(t-3),$$
$$x(0) = x'(0) = 0.$$

Solution - Taking the Laplace transform of both sides of the equation we get

$$s^2X(s) - 4sX(s) + 4X(s) = 1 + e^{-3s}.$$

Solving this for $X(s)$ we get

$$X(s) = \frac{1+e^{-3s}}{s^2-4s+4} = \frac{1+e^{-3s}}{(s-2)^2}.$$

The inverse Laplace transform of $X(s)$ is

$$x(t) = te^{2t} + u(t-3)(t-3)e^{2(t-3)}.$$

6. (30 Points) *Power Series Solutions*

Find the general solution to the ordinary differential equation

$$y'' + xy' + y = 0$$

using power series methods.

Note - The product of all the odd numbers up to and including $(2n + 1)$ is written $(2n + 1)!!$. The product of all even numbers up to and including $(2n)$ is $2^n(n!)$.

Solution - If we write $y(x) = \sum_{n=0}^{\infty} c_n x^n$ and plug this power series into the differential equation we get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Shifting the first summation by 2 we get

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n =$$
$$\sum_{n=0}^{\infty}[(n+2)(n+1)c_{n+2} + (n+1)c_n]x^n = 0.$$

For this to be true, we must have the recurrence relation

$$c_{n+2} = -\frac{c_n}{n+2}.$$

For the even terms we get

$$c_0 = c_0,$$
$$c_2 = -\frac{c_0}{2},$$
$$c_4 = -\frac{c_2}{4} = \frac{c_0}{4 \times 2},$$

... and in general ...

$$c_{2n} = \frac{(-1)^n}{2^n(n!)} c_0.$$

For the odd terms we have

$$c_1 = c_1,$$
$$c_3 = -\frac{c_1}{3},$$
$$c_5 = -\frac{c_3}{5} = \frac{c_1}{5 \times 3 \times 1},$$

... and in general ...

$$c_{2n+1} = \frac{(-1)^n}{(2n+1)!!} c_1.$$

So, our general solution is

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n!)} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!!} x^{2n+1}.$$