Math 2280 - Exam 2

University of Utah

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This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (5 points) *System Conversion*

   Convert the following linear differential equation

$$x^{(4)} - 3x^{(3)} + 12x'' - 15x' + 8x = e^t$$

   into an equivalent system of first-order linear differential equations.

   Solution - We define the variables

$$x = x_1, x_2 = x_1', x_3 = x_2', x_4 = x_3'.$$

   From these, we get the equivalent system of first-order linear differential equations:

$$x_1' = x_2;$$

$$x_2' = x_3;$$

$$x_3' = x_4;$$

$$x_4' = 3x_4 - 12x_3 + 15x_2 - 8x_1 + e^t.$$

2. (10 points) *Particular Solutions*

   Using the method of undetermined coefficients, provide the form of the particular solution to the linear differential equation

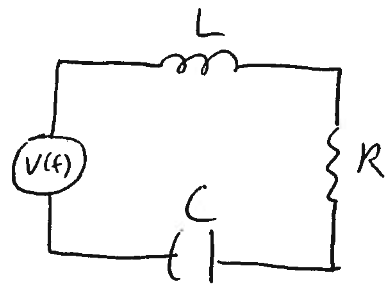
$$x^{(3)} - 6x'' + 11x' - 6x = t^2 e^t \sin(2t).$$

   You do not need to solve for the undetermined coefficients.

   Solution - $x_p = (At^2 + Bt + C)e^t \sin(2t) + (Dt^2 + Et + F)e^t \cos(2t)$.

3. (40 points) *LRC Circuits*

For the LRC circuit:



with $V(t) = 13\cos(2t)$, $L = 1$, $R = 2$, and $C = 1$

(a) (5 points) What is the second-order linear differential equation that models the behavior of this system?

Solution -

$$Q''(t) + 2Q'(t) + Q(t) = 13\cos(2t).$$

(b) (10 points) Find a a particular solution to this differential equation.

Solution - The form of the particular solution will be:

$$Q_p(t) = A\cos(2t) + B\sin(2t).$$

Plugging this form into our differential equation we get:

$$(4B - 3A)\cos(2t) - (3B + 4A)\sin(2t) = 13\cos(2t).$$

Solving for A and B we get $A = -\frac{39}{25}$ and $B = \frac{52}{25}$. So, our particular solution is

$$Q_p(t) = \frac{52}{25}\sin(2t) - \frac{39}{25}\cos(2t).$$

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(c) (10 points) What is the general solution to this differential equation?

Solution - The characteristic equation for the homogeneous equation is

$$r^2 + 2r + 1 = (r+1)^2.$$

So, $r = -1$ is a root of multiplicity 2, and the corresponding homogeneous solution is:

$$Q_h(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

Therefore, the general solution is:

$$Q(t) = Q_p(t) + Q_h(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{52}{25}\sin(2t) - \frac{39}{25}\cos(2t).$$

(d) (5 points) Is the system overdamped, underdamped, or critically damped?

Solution - The homogeneous solution has a repeated real root, so the system is *critically damped*.

(e) (10 points) What is the current $I(t)$ in the circuit if $I(0) = 5$ and $I'(0) = 8$.

Solution - Our general solution is

$$Q(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{52}{25}\sin(2t) - \frac{39}{25}\cos(2t).$$

The corresponding functions for $I(t) = Q'(t)$ and $I'(t)$ are:

$$I(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} + \frac{104}{25}\cos(2t) + \frac{78}{25}\sin(2t),$$

$$I'(t) = c_1 e^{-t} - 2c_2 e^{-t} + c_2 t e^{-t} - \frac{208}{25}\sin(2t) + \frac{156}{25}\cos(2t).$$

Plugging in $t = 0$ we get:

$$I(0) = -c_1 + c_2 + \frac{104}{25} = 5,$$

$$I'(0) = c_1 - 2c_2 + \frac{156}{25} = 8.$$

Solving these for c_1 and c_2 we get $c_1 = -\frac{86}{25}$ and $c_2 = -\frac{13}{5}$. So, the solution to this initial value problem is:

$$I(t) = \frac{21}{25}e^{-t} + \frac{13}{5}te^{-t} + \frac{104}{25}\cos(2t) + \frac{78}{25}\sin(2t).$$

4. (25 points) *Linear Systems of Differential Equations*

Find the general solution to the system of differential equations:

$$\begin{array}{ccccccc} x_1' & = & 3x_1 & + & x_2 & + & x_3 \\ x_2' & = & -5x_1 & - & 3x_2 & - & x_3 \\ x_3' & = & 5x_1 & + & 5x_2 & + & 3x_3 \end{array}$$

Hint - One of the eigenvalues is $\lambda = 3$.

Solution - To find the solutions to this system we want to find the eigenvalues of the matrix, which are the roots of the characteristic equation:

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ -5 & -3-\lambda & -1 \\ 5 & 5 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} -3-\lambda & -1 \\ 5 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} -5 & -1 \\ 5 & 3-\lambda \end{vmatrix} + 1\begin{vmatrix} -5 & -3-\lambda \\ 5 & 5 \end{vmatrix}$$

$$= (3-\lambda)(\lambda^2 - 4) - (5\lambda - 10) + (5\lambda - 10) = (3-\lambda)(\lambda^2 - 4)$$

$$= (3-\lambda)(\lambda - 2)(\lambda + 2).$$

So, the eigenvalues are $\lambda = 3, 2, -2$. The corresponding eigenvectors are:

For $\lambda = 3$:

$$\begin{pmatrix} 0 & 1 & 1 \\ -5 & -6 & -1 \\ 5 & 5 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From which we get the eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$:

$$\begin{pmatrix} 1 & 1 & 1 \\ -5 & -5 & -1 \\ 5 & 5 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

From which we get the eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Finally, for $\lambda = -2$:

$$\begin{pmatrix} 5 & 1 & 1 \\ -5 & -1 & -1 \\ 5 & 5 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

From these we get the general solution

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-2t}.$$

5. (20 points) *Multiple Eigenvalue Systems*[1]

Find the general solution to the system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix} \mathbf{x}$$

Solution - The characteristic polynomial for this matrix is:

$$\begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda) + 9 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2.$$

So, the eigenvalues are $\lambda = 4, 4$. The corresponding eigenvector equation for this repeated eigenvalue is

$$\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From this we see the only eigenvector is $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and its non-zero multiples. So, we need to form a length 2 chain of generalized eigenvectors.

If we square $(A - 4I)$ we get:

$$\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So, any vector $\mathbf{v}$ that is not itself an eigenvector (or $\mathbf{0}$) will work for $\mathbf{v}_2$. Take

[1] The title of this problem is a hint.

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The corresponding eigenvector $\mathbf{v}_1$ is:

$$\mathbf{v}_1 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}.$$

So, the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^{4t} + c_2 \left(\begin{pmatrix} -3 \\ 3 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{4t} \right).$$