# Math 2280 - Exam 1

University of Utah

Spring 2014

**Name:** Solutions by Dylan Zwick

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (15 Points) *Differential Equation Basics*

**(a)** (5 points) What is the order of the differential equation given below?[1]

$$x^2y^3y^{(4)} - e^{x^2}y^2y' + 3x^{y''} = 42$$

*Solution* - Fourth Order.

**(b)** (5 points) Is the differential equation given below linear?

$$y^{(3)} + y^3 = x + 3$$

*Solution* - No.

**(c)** (5 points) On what intervals are we guaranteed a unique solution exists for the differential equation below?

$$(x+1)y' + e^{2x}y = \frac{x^2+2}{x-1}$$

*Solution* - We can rewrite the differential equation as:

$$y' + \frac{e^{2x}}{(x+1)}y = \frac{x^2+2}{(x+1)(x-1)}.$$

Where guaranteed a unique solution exists on any interval where both $\frac{e^{2x}}{(x+1)}$ and $\frac{x^2+2}{(x+1)(x-1)}$ are continuous. These are the intervals $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$.

[1] Extra credit - Solve this differential equation! Just kidding. Do not attempt to solve it.

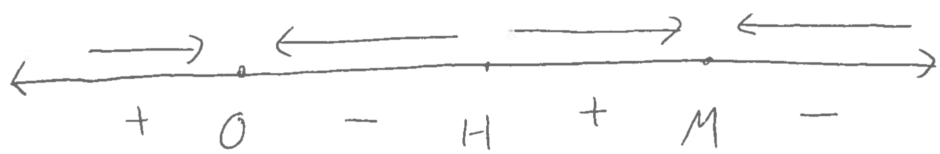
2. (10 points) *Phase Diagrams*

Find the critical points for the autonomous equation:

$$\frac{dP}{dt} = kP(P - M)(H - P),$$

where $k, M, H > 0$ and $M > H$. Draw the corresponding phase diagram, and indicate if the critical points are stable, unstable, or semistable.

*Solution* - The critical points are the values of $P$ for which $\frac{dP}{dt} = 0$. These are the points $P = 0, H, M$. The corresponding phase diagram is:




From the phase diagram we see that $P = 0, M$ are stable critical points, while $P = H$ is unstable.

3. (20 Points) *Separable Equations*

Solve the initial value problem

$$\frac{dP}{dt} = 5P^2,$$

where $P(0) = 2$.

Why would this be an example of a "doomsday" equation? According to this differential equation, when is "doomsday"?

*Solution* - We can rewrite this differential equation as

$$\frac{dP}{P^2} = 5dt.$$

Integrating both sides we get

$$-\frac{1}{P} = 5t + C.$$

Solving for $P$ we get

$$P(t) = \frac{1}{C - 5t}.$$

Plugging in the initial condition $P(0) = 2$ we get $C = \frac{1}{2}$, and so

$$P(t) = \frac{2}{1 - 10t}.$$

This is called a "doomsday" equation because at $t = \frac{1}{10}$ we have a vertical asymptote. So, $t = \frac{1}{10}$ is "doomsday" for this equation.

4. (35 points) *Linear Differential Equations*

(a) (20 points) Solve the initial value problem

$$y' + 4xy - 3y + 3e^{-2x^2} = 0,$$
$$y(0) = 4.$$

*Solution* - We can rewrite this equation as

$$y' + (4x - 3)y = -3e^{-2x^2}.$$

This is a first-order linear differential equation. The integrating factor is

$$\rho(x) = e^{\int (4x-3)dx} = e^{2x^2-3x}.$$

Multiplying both sides by this integrating factor we get

$$e^{2x^2-3x}y' + (4x-3)e^{2x^2-3x}y = \frac{d}{dx}\left(e^{2x^2-3x}y\right) = -3e^{-3x}.$$

Integrating both sides we get

$$e^{2x^2-3x}y = e^{-3x} + C$$
$$\Rightarrow y(x) = e^{-2x^2} + Ce^{-2x^2+3x}.$$

Plugging in the initial condition we get

$$4 = y(0) = 1 + C \Rightarrow C = 3.$$

So, the unique solution to the initial value problem is

$$y(x) = e^{-2x^2} + 3e^{-2x^2+3x}.$$

(b) (15 points) Find the general solution to the differential equation

$$y'' - y' - 6y = 0.$$

*Solution* - The corresponding characteristic equation is

$$r^2 - r - 6 = (r-3)(r+2).$$

So, the roots are $r = -2, 3$, and the general solution to the differential equation is

$$y(x) = c_1 e^{3x} + c_2 e^{-2x}.$$

5. (20 points) *Euler's Method*

Use Euler's method with step size $h = 1$ to estimate $y(2)$, where $y(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = 2x + 3y,$$

$$y(0) = 2.$$

*Solution* - Using Euler's method our estimate for $y(1)$ is

$$y(1) \approx 2 + (2(0) + 3(2)) = 8.$$

Our estimate for $y(2)$ is

$$y(2) \approx 8 + (2(1) + 3(8)) = 34.$$