Math 2280 - Assignment 9

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Section 7.1 - 1, 6, 20, 30, 36

Section 7.2 - 1, 4, 15, 20, 29

Section 7.3 - 3, 8, 19, 24, 30, 33

Section 7.1 - Laplace Transforms and Inverse Transforms

7.1.1 - Calculate the Laplace transform of $f(t) = t$ using the definition of the Laplace transform.

Solution - The Laplace transform is:

$$\mathcal{L}(f(t)) = \int_0^\infty te^{-st}dt$$

$$= -\frac{te^{-st}}{s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}.$$

The move from the first line to the second requires an integration by parts. If $s > 0$ then the limits above converge, and we get:

$$= -\frac{e^{-st}}{s^2}\bigg|_0^\infty = \frac{1}{s^2}.$$

7.1.6 - Calculate the Laplace transform of $f(t) = \sin^2 t$ using the definition of the Laplace transform.

Solution - The Laplace transform is:

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st}\sin^2(t)dt$$
$$= -\frac{e^{-st}\sin^2(t)}{s}\Big|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\sin(t)\cos(t)dt.$$

Again, the move from the first line to the second required an integration by parts, and if $s > 0$ the limits above converge and we get (after another integration by parts):

$$-\frac{2\sin(t)\cos(t)e^{-st}}{s^2}\Big|_0^\infty + \frac{2}{s^2}\int_0^\infty e^{-st}(\cos^2(t) - \sin^2(t))dt.$$

Noting

$$\cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t)$$

we get:

$$= \frac{2}{s^2}\int_0^\infty e^{-st}(1 - 2\sin^2(t))dt.$$

Combining the string of equalities above we get:

$$\left(1 + \frac{4}{s^2}\right)\int_0^\infty e^{-st}\sin^2(t)dt = \frac{2}{s^2}\int_0^\infty e^{-st}dt = \frac{2}{s^3}.$$

So,

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st}\sin^2(t)dt = \left(\frac{s^2}{s^2+4}\right)\left(\frac{2}{s^3}\right) = \frac{2}{s(s^2+4)}.$$

7.1.20 - Find the Laplace transform of the function $f(t) = te^t$.

Solution - Using the definition of the Laplace transform we get:

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st}te^t dt = \int_0^\infty te^{(1-s)t}dt.$$

After an integration by parts this becomes:

$$\left.\frac{te^{(1-s)t}}{1-s}\right|_0^\infty - \frac{1}{1-s}\int_0^\infty e^{(1-s)t}dt.$$

If $s > 1$ the limits converge, and we get:

$$= -\left.\frac{e^{(1-s)t}}{(1-s)^2}\right|_0^\infty = \frac{1}{(1-s)^2}.$$

7.1.30 - Find the inverse Laplace transform of the function $F(s) = \dfrac{9+s}{4-s^2}$.

Solution - If we break up this fraction we get:

$$\mathcal{L}^{-1}\left(\frac{9+s}{4-s^2}\right) = -\frac{9}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2-4}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2-4}\right)$$
$$= -\frac{9}{2}\sinh(2t) - \cosh(2t).$$

7.1.36 - Show that the function $f(t) = \sin(e^{t^2})$ is of exponential order as $t \to \infty$ but that its derivative is not.

Solution - We have

$$\sin(e^{t^2}) \leq 1 < e^t$$

for $t > 0$. So, $f(t)$ is of exponential order with $M = 1, c = 1, T = 0$.

On the other hand the derivative is $f(t)$ is $f'(t) = 2te^{t^2}\cos(e^{t^2})$. Suppose there exist positive constants M, c and T such that:

$$2te^{t^2}\cos(e^{t^2}) < Me^{ct}$$

for all $t > T$. Now, for any positive c we know $e^{t^2} > e^{ct}$ when $t > c$, and in fact $e^{t^2} > Me^{ct}$ when $t > \dfrac{c + \sqrt{c^2 + 4\ln(M)}}{2}$.[1] Furthermore, the cosine function is periodic with maximum 1, so for any positive T there exists a $t > T$ such that

$$2te^{t^2}\cos(e^{t^2}) = 2te^{t^2}.$$

And, for $t > \dfrac{1}{2}$ we have $e^{t^2} < 2te^{t^2}$. So, we can see that, no matter what values of M, c, and T we pick there will always be a $t > T$ such that $f'(t) > Me^{ct}$, which means $f'(t)$ is not of exponential order.

[1] We can assume $M > 1$.

Section 7.2 - Transformation of Initial Value Problems

7.2.1 - Use Laplace transforms to solve the initial value problem below.

$$x'' + 4x = 0; \qquad x(0) = 5; x'(0) = 0.$$

Solution - The Laplace transform of the left side is:

$$\mathcal{L}(x'' + 4x) = s^2X(s) - sx(0) - x'(0) + 4X(s)$$
$$= s^2X(s) - 5s + 4X(s) = (s^2+4)X(s) - 5s.$$

The Laplace transform of the right side is 0, and equating the Laplace transforms we get:

$$(s^2+4)X(s) - 5s = 0$$
$$\Rightarrow X(s) = \frac{5s}{s^2+4} = 5\left(\frac{s}{s^2+4}\right).$$

From this we see the inverse Laplace transform, our solution, is:

$$x(t) = 5\cos(2t).$$

7.2.4 - Use Laplace transforms to solve the initial value problem below.

$$x'' + 8x' + 15x = 0; \qquad x(0) = 2; x'(0) = -3.$$

Solution - The Laplace transform of the left side of the above equation is:

$$\mathcal{L}(x'' + 8x' + 15x) = s^2X(S) - sx(0) - x'(0) + 8sX(s) - 8x(0) + 15X(s)$$

$$= (s^2 + 8s + 15)X(s) - 2s - 13.$$

The Laplace transform of the right side is 0, and equating these two Laplace transforms we get:

$$(s^2 + 8s + 15)X(s) - 2s - 13 = 0$$

$$\Rightarrow X(s) = \frac{2s+13}{s^2+8s+15} = \frac{2s+13}{(s+5)(s+3)}.$$

If we take a partial fraction decomposition of the rational function we get:

$$\frac{2s+13}{(s+5)(s+3)} = \frac{A}{s+5} + \frac{B}{s+3} == \frac{A(s+3)+B(s+5)}{(s+5)(s+3)}.$$

So, we get the linear equations:

$$A + B = 2$$

$$3A + 5B = 13.$$

Solving for A and B we get $A = -\frac{3}{2}, B = \frac{7}{2}$. So,

$$X(s) = -\frac{3}{2}\left(\frac{1}{s+5}\right) + \frac{7}{2}\left(\frac{1}{s+3}\right).$$

From this we get the inverse Laplace transform, our solution, is:

$$x(t) = -\frac{3}{2}e^{-5t} + \frac{7}{2}e^{-3t}.$$

7.2.15 - Use Laplace transforms to solve the initial value problem below.

$$x'' + x' + y' + 2x - y = 0,$$
$$y'' + x' + y' + 4x - 2y = 0;$$
$$x(0) = y(0) = 1; \qquad x'(0) = y'(0) = 0.$$

Solution - If we take the Laplace transforms of both equations we get:

$$s^2X(s) - s + sX(s) - 1 + sY(s) - 1 + 2X(s) - Y(s) = 0,$$
$$s^2Y(s) - s + sX(s) - 1 + sY(s) - 1 + 4X(s) - 2Y(s) = 0.$$

We can simplify this system as:

$$(s^2 + s + 2)X(s) + (s-1)Y(s) = s + 2,$$
$$(s+4)X(s) + (s^2 + s - 2)Y(s) = s + 2.$$

Solving for for $Y(s)$ in the second equation we get:

$$Y(s) = \frac{s+2}{s^2+s-2} - \frac{(s+4)X(s)}{s^2+s-2} = \frac{1}{s-1} - \frac{(s+4)X(s)}{(s+2)(s-1)}.$$

If we plug this in for $Y(s)$ in the first equation we get:

$$(s^2 + s + 2)X(s) + 1 - \left(\frac{s+4}{s+2}\right)X(s) = s + 2.$$

Solving for $X(s)$ we get:

$$X(s) = \frac{(s+1)(s+2)}{s(s^2+3s+3)}.$$

Plugging this into our equation for $Y(s)$ we get:

$$Y(s) = \frac{1}{s-1} - \frac{(s+4)(s+1)}{s(s^2+3s+3)(s-1)} = \frac{s^3+2s^2-2s-4}{s(s-1)(s^2+3s+3)}.$$

A partial fraction decomposition on $X(s)$ gives us:

$$X(s) = \frac{(s+1)(s+2)}{s(s^2+3s+3)} = \frac{A}{s} + \frac{Bs+C}{s^2+3s+3}.$$

If we solve for A, B, C using our usual procedure we get:

$$X(s) = \frac{2}{3}\left(\frac{1}{s}\right) + \frac{\frac{1}{3}s + \frac{1}{2}}{\left(s+\frac{3}{2}\right)^2 + \frac{3}{4}} + \frac{\frac{1}{2}}{\left(s+\frac{3}{2}\right)^2 + \frac{3}{4}},$$

from which we get:

$$x(t) = \frac{2}{3} + \frac{1}{3}e^{-\frac{3}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

Doing the same thing with $Y(s)$ we get:

$$\begin{aligned} Y(s) &= \frac{s^3+2s^2-2s-4}{s(s-1)(s^2+3s+3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+d}{s^2+3s+3} \\ &= \frac{28}{21}\left(\frac{1}{s}\right) - \frac{9}{21}\left(\frac{1}{s-1}\right) + \frac{\frac{2}{21}s + \frac{15}{21}}{(\left(s+\frac{3}{2}\right)^2 + \frac{3}{4}}. \end{aligned}$$

So,

$$y(t) = \frac{1}{21}\left(28 - 9e^{t} + 2e^{-\frac{3}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + 4\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)\right).$$

Yeah, this one was a pain...

7.2.20 - Apply Theorem 2 from the textbook to find the inverse Laplace transform of the function

$$F(s) = \frac{2s+1}{s(s^2+9)}.$$

Solution - Theorem 2 states

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)d\tau.$$

If we take $G(s) = \frac{2s+1}{s^2+9}$ we can rewrite this as:

$$G(s) = 2\left(\frac{s}{s^2+9}\right) + \frac{1}{3}\left(\frac{3}{s^2+9}\right).$$

The inverse Laplace transform of $G(s)$ is

$$g(t) = 2\cos(3t) + \frac{1}{3}\sin(3t).$$

We note that $F(s) = \frac{G(s)}{s}$, and so applying theorem 2 we get the inverse Laplace transform of $F(s)$ is:

$$f(t) = \int_0^t (2\cos(3\tau) + \frac{1}{3}\sin(3\tau))d\tau = \frac{2}{3}\sin(3\tau) - \frac{1}{9}\cos(3\tau)\Big|_0^t$$

$$= \frac{2}{3}\sin(3t) - \frac{1}{9}\cos(3t) + \frac{1}{9}.$$

7.2.29 - Derive the Laplace transform given below:

$$\mathcal{L}(t \sinh kt) = \frac{2ks}{(s^2 - k^2)^2}$$

Solution - We note:

$$\begin{aligned} f(t) &= t \sinh(kt) \\ f'(t) &= \sinh(kt) + kt \cosh(kt) \\ f''(t) &= 2k \cosh(kt) + k^2 t \sinh(kt). \end{aligned}$$

Now,

$$\mathcal{L}(f''(t)) = s^2 F(s) - sf(0) - f'(0) = s^2 F(s).$$

So,

$$\begin{aligned} 2k\mathcal{L}(\cosh(kt)) + k^2 \mathcal{L}(t \sinh(kt)) &= s^2 \mathcal{L}(t \sinh(kt)) \\ \frac{2k\mathcal{L}(\cosh(kt))}{s^2 - k^2} &= \mathcal{L}(t \sinh(kt)). \end{aligned}$$

As $\mathcal{L}(\cosh(kt)) = \dfrac{s}{s^2 - k^2}$ we get:

$$\mathcal{L}(t \sinh(kt)) = \frac{2ks}{(s^2 - k^2)^2}.$$

Section 7.3 - Translation and Partial Fractions

7.3.3 - Apply the translation theorem to find the Laplace transform of the function

$$f(t) = e^{-2t} \sin 3\pi t.$$

Solution - The translation theorem states

$$\mathcal{L}(e^{at} f(t)) = F(s-a).$$

So,

$$\mathcal{L}(e^{-2t} \sin(3\pi t)) = F(s+2),$$

where

$$F(s) = \mathcal{L}(\sin(3\pi t)) = \frac{3\pi}{s^2 + 9\pi^2}.$$

Therefore,

$$\mathcal{L}(e^{-2t} \sin(3\pi t)) = F(s+2) = \frac{3\pi}{(s+2)^2 + 9\pi^2}.$$

7.3.8 - Apply the translation theorem to find the inverse Laplace transform of the function

$$F(s) = \frac{s+2}{s^2+4s+5}.$$

Solution -

$$F(s) = \frac{s+2}{s^2+4s+5} = \frac{s+2}{(s+2)^2+1} = \tilde{F}(s+2)$$

where

$$\tilde{F}(s) = \frac{s}{s^2+1}.$$

Now,

$$\mathcal{L}^{-1}(\tilde{F}(s)) = \cos(t).$$

So, using the translation theorem

$$\mathcal{L}^{-1}(\tilde{F}(s+2)) = e^{-2t}\cos(t).$$

7.3.19 - Use partial fractions to find the inverse Laplace transform of the function

$$F(s) = \frac{s^2 - 2s}{s^4 + 5s^2 + 4}.$$

Solution - We can factor the denominator of the rational function above as:

$$F(s) = \frac{s^2 - 2s}{s^4 + 5s^2 + 4} = \frac{s^2 - 2s}{(s^2 + 4)(s^2 + 1)}.$$

Taking a partial fraction decomposition we get:

$$\frac{s^2 - 2s}{(s^2 + 4)(s^2 + 1)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 1}.$$

If we solve for these coefficients we get:

$$A = \frac{2}{3}, B = \frac{4}{3}, C = -\frac{2}{3}, D = -\frac{1}{3}.$$

So, our partial fraction decomposition is:

$$F(s) = \frac{1}{3}\left(\frac{2s + 4}{s^2 + 4}\right) + \frac{1}{3}\left(\frac{-2s - 1}{s^2 + 1}\right)$$

$$= \frac{2}{3}\left(\frac{s + 2}{s^2 + 4}\right) - \frac{1}{3}\left(\frac{2s + 1}{s^2 + 1}\right)$$

$$= \frac{2}{3}\left(\frac{s}{s^2 + 4}\right) + \frac{2}{3}\left(\frac{2}{s^2 + 4}\right) - \frac{2}{3}\left(\frac{s}{s^2 + 1}\right) - \frac{1}{3}\left(\frac{1}{s^2 + 1}\right).$$

This has inverse Laplace transform

$$f(t) = \frac{2}{3}\cos(2t) + \frac{2}{3}\sin(2t) - \frac{2}{3}\cos(t) - \frac{1}{3}\sin(t).$$

7.3.24 - Use the factorization

$$s^4 + 4a^4 = (s^2 - 2as + 2a^2)(s^2 + 2as + 2a^2)$$

to derive the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{s}{s^4 + 4a^4}\right\} = \frac{1}{2a^2}\sinh at \sin at.$$

Solution - If we take a partial fraction decomposition of

$$\frac{s}{s^4 + 4a^4}$$

we get

$$\frac{s}{s^4 + 4a^4} = \frac{As + B}{s^2 - 2as + 2a^2} + \frac{Cs + D}{s^2 + 2as + 2a^2}$$

$$= \frac{(A + C)s^3 + (2aA + B - 2aC + D)s^2 + (2a^2A + 2aB + 2a^2C - 2aD)s + (2a^2B + 2a^2D)}{s^4 + 4a^4}.$$

From this we get the system of equations:

$$A + C = 0,$$

$$2aA + B - 2aC + D = 0,$$

$$2a^2A + 2aB + 2a^2C - 2aD = 1,$$

$$2a^2B + 2a^2D = 0.$$

So, $A = -C$ from the first equality. Plugging this into the third equality we get

$$2aB - 2aD = 1$$

while for the fourth equality we have $B = -D$.

Combining these we have $4aB = 1$, which gives us $B = \frac{1}{4a}$, while $D = -\frac{1}{4a}$.

The second equality tells us, given $B = -D$, that $A = C$. As $A = -C$ this implies $A = C = 0$.

So, we get the decomposition

$$\begin{aligned} \frac{s}{s^4 + 4a^4} &= \frac{1}{4a}\left(\frac{1}{s^2 - 2as + 2a^2} - \frac{1}{s^2 + 2as + 2a^2}\right) \\ &= \frac{1}{4a}\left(\frac{1}{(s-a)^2 + a^2} - \frac{1}{(s+a)^2 + a^2}\right). \end{aligned}$$

Taking the inverse Laplace transform we get:

$$\begin{aligned} &\frac{1}{4a}\left(\frac{e^{at}\sin(at)}{a} - \frac{e^{-at}\sin(at)}{a}\right) \\ &= \frac{1}{2a^2}\left(\frac{e^{at} - e^{-at}}{2}\right)\sin(at) \\ &= \frac{1}{2a^2}\sinh(at)\sin(at). \end{aligned}$$

7.3.30 - Use Laplace transforms to solve the initial value problem

$$x'' + 4x' + 8x = e^{-t} \quad x(0) = x'(0) = 0.$$

Solution - The laplace transform of the left side is:

$$\mathcal{L}(x'' + 4x' + 8x) =$$
$$s^2X(s) - sx(0) - x'(0) + 4sX(s) - 4x(0) + 8X(s) = (s^2 + 4s + 8)X(s).$$

The Laplace transform of the right side is:

$$\mathcal{L}(e^{-t}) = \frac{1}{s+1}.$$

Equating the two sides we get:

$$(s^2 + 4s + 8)X(s) = \frac{1}{s+1}$$
$$X(s) = \frac{1}{(s+1)(s^2+4s+8)} = \frac{1}{(s+1)((s+2)^2+4)}.$$

The partial fraction decomposition of this rational function is

$$X(s) = \frac{\frac{1}{5}}{s+1} + \frac{-\frac{1}{5}s - \frac{3}{5}}{(s+2)^2+4)}$$
$$= \frac{1}{5}\left(\frac{1}{s+1}\right) - \frac{1}{5}\left(\frac{s+2}{(s+2)^2+4}\right) - \frac{1}{10}\left(\frac{2}{(s+2)^2+4}\right).$$

Taking the inverse Laplace transform we get:

$$x(t) = \frac{e^{-t}}{5} - \frac{e^{-2t}\cos(2t)}{5} - \frac{e^{-2t}\sin(2t)}{10}.$$

7.3.33 - Use Laplace transforms to solve the initial value problem

$$x^{(4)} + x = 0 \quad x(0) = x'(0) = x''(0) = 0, x^{(3)}(0) = 1.$$

The Laplace transform of the left side of the equation above is

$$\begin{aligned}\mathcal{L}(x^{(4)} + x) &= s^4 X(s) - s^3 x(0) - s^2 x'(0) - s x''(0) - x^{(3)}(0) + X(s) \\ &= s^4 X(s) - 1 + X(s).\end{aligned}$$

The Laplace transform of the right side is 0. Equating the two sides we get:

$$\begin{aligned}X(s) &= \frac{1}{s^4+1} = \frac{1}{(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)} \\ &= \frac{As+B}{s^2+\sqrt{2}s+1} + \frac{Cs+D}{s^2-\sqrt{2}s+1}.\end{aligned}$$

A partial fraction decomposition gives us

$$A = \frac{\sqrt{2}}{4}, B = \frac{1}{2}, C = -\frac{\sqrt{2}}{4}, D = \frac{1}{2}.$$

From these we get:

$$X(s) = \frac{\sqrt{2}}{4}\left(\frac{s+\frac{1}{\sqrt{2}}}{\left(s+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right) + \frac{\sqrt{2}}{4}\left(\frac{\frac{1}{\sqrt{2}}}{\left(s+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right) - \frac{\sqrt{2}}{4}\left(\frac{s-\frac{1}{\sqrt{2}}}{\left(s-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right) + \frac{\sqrt{2}}{4}\left(\frac{\frac{1}{\sqrt{2}}}{\left(s-\frac{1}{\sqrt{2}}\right)2+\frac{1}{2}}\right).$$

The inverse Laplace transform of this will be:

$$x(t) = \frac{\sqrt{2}}{4}e^{-\frac{t}{\sqrt{2}}}\cos\left(\frac{t}{\sqrt{2}}\right) + \frac{\sqrt{2}}{4}e^{-\frac{t}{\sqrt{2}}}\sin\left(\frac{t}{\sqrt{2}}\right) - \frac{\sqrt{2}}{4}e^{-\frac{t}{\sqrt{2}}}\cos\left(\frac{t}{\sqrt{2}}\right) + \frac{\sqrt{2}}{4}e^{-\frac{t}{\sqrt{2}}}\sin\left(\frac{t}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cosh\left(\frac{t}{\sqrt{2}}\right)\sin\left(\frac{t}{\sqrt{2}}\right) - \sinh\left(\frac{t}{\sqrt{2}}\right)\cos\left(\frac{t}{\sqrt{2}}\right)\right).$$