Math 2280 - Assignment 8

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Section 5.5 - 1, 7, 9, 18, 24

Section 5.6 - 1, 6, 10, 17, 19

Matrix Exponentials and Linear Systems

5.5.1 - Find a fundamental matrix for the system below, and then find a solution satisfying the given initial conditions.

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Solution - First we find the eigenvalues of the coefficient matrix:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

So, the eigenvalues are $\lambda = 3, 1$.

For the eigenvalue $\lambda = 3$ an eigenvector must satisfy:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and we see the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works.

For the eigenvalue $\lambda = 1$ an eigenvector must satisfy:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and we see the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works.

So, we get the two linearly independent solutions:

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t,$$

$$\mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

A corresponding funadmental matrix is

$$\Phi(t) = \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix}.$$

To find a solution satisfying the given initial conditions we use

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

which has determinant 2, and inverse

$$\Phi(0)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

From this we get

$$\mathbf{c} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix}.$$

So, a solution satisfying the initial conditions is:

$$\mathbf{x}(t) = \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix} \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2}e^t + \frac{1}{2}e^{3t} \\ -\frac{5}{2}e^t + \frac{1}{2}e^{3t} \end{pmatrix}.$$

5.5.7 - Find a fundamental matrix for the system below, and then find a solution satisfying the given initial conditions.

$$\mathbf{x}' = \begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Solution - First, we find the eigenvalues of the coefficient matrix:

$$\begin{vmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{vmatrix}$$

$$= (5-\lambda)[(-1-\lambda)(-4-\lambda) - (-2)(-2)] + (-6)[(2)(-2) - (-1-\lambda)(4)]$$

$$= \lambda - \lambda^3 = \lambda(1+\lambda)(1-\lambda).$$

So, the eigenvalues are $\lambda = 0, -1, 1$.

For $\lambda = 0$ an eigenvector must satisfy:

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and we see $\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$ works.

For $\lambda = 1$ an eigenvector must satisfy:

$$\begin{pmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and we see $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ works.

For $\lambda = -1$ an eigenvector must satisfy:

$$\begin{pmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and we see $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ works.

So, a fundamental matrix for this system is:

$$\Phi(t) = \begin{pmatrix} 6 & 3e^t & 2e^{-t} \\ 2 & e^t & e^{-t} \\ 5 & 2e^t & 2e^{-t} \end{pmatrix}.$$

From this we get:

$$\Phi(0) = \begin{pmatrix} 6 & 3 & 2 \\ 2 & 1 & 1 \\ 5 & 2 & 2 \end{pmatrix},$$

$$\Phi(0)^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & 3 & 0 \end{pmatrix}.$$

Using our initial conditions we get:

$$\mathbf{c} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \\ -1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}.$$

So, the solution that satisfies our initial conditions is:

$$\mathbf{x}(t) = \begin{pmatrix} 6 & 3e^t & 2e^{-5} \\ 2 & e^t & e^{-t} \\ 5 & 2e^t & 2e^{-t} \end{pmatrix} \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 + 12e^t + 2e^{-t} \\ -4 + 4e^t + e^{-t} \\ -10 + 8e^t + 2e^{-t} \end{pmatrix}.$$

5.5.9 - Compute the matrix exponential $e^{\mathbf{A}t}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\begin{array}{rcrcr} x_1' & = & 5x_1 & - & 4x_2 \\ x_2' & = & 2x_1 & - & x_2 \end{array}$$

Solution - First we calculate the eigenvalues of the coefficient matrix:

$$\left| \begin{array}{cc} 5-\lambda & -4 \\ 2 & -1-\lambda \end{array} \right| = (5-\lambda)(-1-\lambda) - (-4)(2)$$

$$= \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

So, the eigenvalues are $\lambda = 3, 1$.

The eigenvector for $\lambda = 3$ must satisfy:

$$\left(\begin{array}{cc} 2 & -4 \\ 2 & -4 \end{array} \right) \mathbf{v} = \left(\begin{array}{c} 0 \\ 0 \end{array} \right).$$

The vector $\left(\begin{array}{c} 2 \\ 1 \end{array} \right)$ works.

For the eigenvalue $\lambda = 1$ the eigenvector must satisfy:

$$\left(\begin{array}{cc} 4 & -4 \\ 2 & -2 \end{array} \right) \mathbf{v} = \left(\begin{array}{c} 0 \\ 0 \end{array} \right).$$

The vector $\left(\begin{array}{c} 1 \\ 1 \end{array} \right)$ works.

So, a fundamental matrix for this system is:

$$\Phi(t) = \begin{pmatrix} 2e^{3t} & e^{t} \\ e^{3t} & e^{t} \end{pmatrix}.$$

For this fundamental matrix we have:

$$\Phi(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\Phi(0)^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

So,

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} 2e^{3t} & e^{t} \\ e^{3t} & e^{t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{3t} - e^{t} & -2e^{3t} + 2e^{t} \\ e^{3t} - e^{t} & -e^{3t} + 2e^{t} \end{pmatrix}.$$

5.5.18 - Compute the matrix exponential $e^{\mathbf{A}t}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ below.

$$\begin{array}{rcrcr} x_1' & = & 4x_1 & + & 2x_2 \\ x_2' & = & 2x_1 & + & 4x_2 \end{array}$$

Solution - First, we calculate the eigenvalues of the coefficient matrix:

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 4 = \lambda^2 - 8\lambda + 12 = (\lambda - 6)(\lambda - 2).$$

So, the eigenvalues are $\lambda = 2, 6$.

An eigenvector for $\lambda = 2$ must satisfy

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and we see $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works.

For the eigenvalue $\lambda = 6$ an eigenvector must satisfy

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ works.

So, we have a fundamental matrix:

$$\Phi(t) = \begin{pmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{pmatrix}.$$

From this fundamental matrix we get:

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\Phi(0)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

So,

$$e^{At} = \begin{pmatrix} e^{2t} & e^{6t} \\ -e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{2t} + \frac{1}{2}e^{6t} & -\frac{1}{2}e^{2t} + \frac{1}{2}e^{6t} \\ -\frac{1}{2}e^{2t} + \frac{1}{2}e^{6t} & \frac{1}{2}e^{2t} + \frac{1}{2}e^{6t} \end{pmatrix}.$$

5.5.24 - Show that the matrix **A** is nilpotent and then use this fact to find the matrix exponential $e^{\mathbf{A}t}$.

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix}$$

Solution - The powers of the matrix are:

$$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 36 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, the matrix is nilpotent, and the matrix exponential is:

$$e^{At} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{pmatrix} t + \begin{pmatrix} 0 & 0 & 0 \\ 36 & 0 & -36 \\ 0 & 0 & 0 \end{pmatrix} \frac{t^2}{2}$$

$$= \begin{pmatrix} 1+3t & 0 & -3t \\ 5t+18t^2 & 1 & 7t-18t^2 \\ 3t & 0 & 1-3t \end{pmatrix}.$$

Nonhomogeneous Linear Systems

5.6.1 - Apply the method of undetermined coefficients to find a particular solution to the system below.

$$\begin{array}{rcrcrcr} x' & = & x & + & 2y & + & 3 \\ y' & = & 2x & + & y & - & 2 \end{array}$$

Solution - The non-homogeneous part of this system is a constant vector, so we'd "guess" our particular solution is also a constant vector:

$$\mathbf{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

$$\mathbf{x}_p' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Plugging these into the system of differential equations above we get:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

So, a_1, a_2 must satisfy:

$$a_1 + 2a_2 + 3 = 0,$$

$$2a_1 + a_2 - 2 = 0.$$

If we solve this system we get $a_1 = \dfrac{7}{3}, a_2 = -\dfrac{8}{3}$. So, our particular solution is:

$$\mathbf{x}_p = \begin{pmatrix} \frac{7}{3} \\ -\frac{8}{3} \end{pmatrix}.$$

5.6.6 - Apply the method of undetermined coefficients to find a particular solution to the system below.

$$\begin{array}{rcrcrcl} x' & = & 9x & + & y & + & 2e^t \\ y' & = & -8x & - & 2y & + & te^t \end{array}$$

Solution - The non-homogeneous term is:

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} te^t,$$

so we'd "guess" the particular solution is of the form:

$$\mathbf{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^t,$$

$$\mathbf{x}_p' = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} e^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^t.$$

Plugging these into the system of differential equations we get:

$$\begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} e^t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^t =$$
$$\begin{pmatrix} 9a_1 + a_2 + 2 \\ -8a_1 - 2a_2 \end{pmatrix} e^t + \begin{pmatrix} 9b_1 + b_2 \\ -8b_1 - 2b_2 + 1 \end{pmatrix} te^t.$$

So, we get the four equations with four unknowns:

$$9a_1 + a_2 + 2 = a_1 + b_1,$$

$$-8a_1 - 2a_2 = a_2 + b_2,$$

$$9b_1 + b_2 = b_1,$$

$$-8b_1 - 2b_2 + 1 = b_2.$$

If we solve this system we get $a_1 = -\frac{91}{256}, a_2 = \frac{25}{32}, b_1 = -\frac{1}{16}, b_2 = \frac{1}{2}$.
So, our particular solution is:

$$\mathbf{x}_p = \frac{1}{256}\begin{pmatrix} -91 \\ 200 \end{pmatrix} e^t + \frac{1}{16}\begin{pmatrix} -1 \\ 8 \end{pmatrix} te^t.$$

5.6.10 - Apply the method of undetermined coefficients to find a particular solution to the system below.

$$\begin{array}{rcrcrcl} x' & = & x & - & 2y & & \\ y' & = & 2x & - & y & + & e^t \sin t \end{array}$$

Solution - The non-homogeneous term in our system of differential equations is:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \sin(t).$$

So, our "guess" for the particular solution will be:

$$\mathbf{x}_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t \sin(t) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^t \cos(t).$$

$$\mathbf{x}_p' = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix} e^t \sin(t) + \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} e^t \cos(t).$$

Plugging these into the given system of differential equations we get:

$$\begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix} e^t \sin(t) + \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} e^t \cos(t) =$$
$$\begin{pmatrix} a_1 - 2a_2 \\ 2a_1 - a_2 + 1 \end{pmatrix} e^t \sin(t) + \begin{pmatrix} b_1 - 2b_2 \\ 2b_1 - b_2 \end{pmatrix} e^t \cos(t).$$

From this we get the four equations:

$$a_1 - b_1 = a_1 - 2a_2,$$

$$a_2 - b_2 = 2a_1 - a_2 + 1,$$

$$a_1 + b_2 = b_1 - 2b_2,$$

$$a_2 + b_2 = 2b_1 - b_2.$$

Solving this system we get $a_1 = -\frac{6}{13}, a_2 = \frac{2}{3}, b_1 = \frac{4}{13}, b_2 = \frac{2}{13}$. So, our particular solution is:

$$\mathbf{x}_p = \frac{1}{13}\begin{pmatrix} -6 \\ 2 \end{pmatrix} e^t \sin(t) + \frac{1}{13}\begin{pmatrix} 4 \\ 3 \end{pmatrix} e^t \cos(t).$$

5.6.17 - Use the method of variation of parameters to solve the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t),$$

$$\mathbf{x}(a) = \mathbf{x}_a.$$

The matrix exponential $e^{\mathbf{A}t}$ is given.

$$\mathbf{A} = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 60 \\ 90 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$e^{\mathbf{A}t} = \frac{1}{6}\begin{pmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{pmatrix}$$

Solution - To solve this we first want to calculate the integral:

$$\int_0^t e^{-As}\mathbf{f}(s)ds = \int_0^t \frac{1}{6}\begin{pmatrix} -e^{s} + 7e^{-5s} & 7e^{s} - 7e^{-5s} \\ -e^{s} + e^{-5s} & 7e^{s} - e^{-5s} \end{pmatrix}\begin{pmatrix} 60 \\ 90 \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} 95e^{s} - 35e^{-5s} \\ 95e^{s} - 5e^{-5s} \end{pmatrix} ds = \begin{pmatrix} 95e^{s} + 7e^{-5s} \\ 95e^{s} + e^{-5s} \end{pmatrix}\Bigg|_0^t$$

$$= \begin{pmatrix} 95e^{t} + 7e^{-5t} - 102 \\ 95e^{t} + e^{-5t} - 96 \end{pmatrix}.$$

From this we get the solution:

$$e^{At}\mathbf{x}_0 + e^{At}\int_0^t e^{-As}\mathbf{f}(s)ds$$

$$= \frac{1}{6}\begin{pmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} +$$

$$\frac{1}{6}\begin{pmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{pmatrix}\begin{pmatrix} 95e^{t} + 7e^{-5t} - 102 \\ 95e^{t} + e^{-5t} - 96 \end{pmatrix}$$

$$= \begin{pmatrix} 102 - 95e^{-t} - 7e^{5t} \\ 96 - 95e^{-t} - e^{5t} \end{pmatrix}.$$

5.6.19 - Use the method of variation of parameters to solve the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t),$$

$$\mathbf{x}(a) = \mathbf{x}_a.$$

The matrix exponential $e^{\mathbf{A}t}$ is given.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 180t \\ 90 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$e^{\mathbf{A}t} = \frac{1}{5}\begin{pmatrix} e^{-3t} + 4e^{2t} & -2e^{-3t} + 2e^{2t} \\ -2e^{-3t} + 2e^{2t} & 4e^{-3t} + e^{2t} \end{pmatrix}.$$

Solution - First, we note:[1]

$$e^{-A(s-t)} = \frac{1}{5}\begin{pmatrix} e^{3(s-t)} + 4e^{-2(s-t)} & -2e^{3(s-5)} + 2e^{-2(s-t)} \\ -2e^{3(s-t)} + 2e^{-2(s-t)} & 4e^{3(s-t)} + e^{-2(s-t)} \end{pmatrix}.$$

As $e^{At}\mathbf{x}_0 = \mathbf{0}$ because $\mathbf{x}_0 = \mathbf{0}$, we have:

$$\mathbf{x}(t) = \int_0^t e^{-A(s-t)}\mathbf{f}(s)ds$$

$$= \int_0^t \begin{pmatrix} e^{3(s-t)} + 4e^{-2(s-t)} & -2e^{3(s-t)} + 2e^{-2(s-t)} \\ -2e^{3(s-t)} + 2e^{-2(s-t)} & 4e^{3(s-t)} + e^{-2(s-t)} \end{pmatrix}\begin{pmatrix} 36s \\ 18 \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} e^{-3t}(36s - 16)e^{3s} + e^{2t}(144s + 36)e^{-2s} \\ e^{-3t}(-72s + 72)e^{3s} + e^{2t}(72s + 18)e^{-2s} \end{pmatrix} ds$$

[1] I'm doing this in a slightly different way than I did Problem 5.6.17. Fundamentally the two methods are the same, but I wanted you to see both variations.

$$= \left(\begin{array}{c} e^{-3t}(12se^{3s} - 16e^{3s}) + e^{2t}(-72se^{-2s} - 54e^{-2s}) \\ e^{-3t}(-24se^{3s} + 32e^{3s}) + e^{2t}(-36se^{-2s} - 27e^{-2s}) \end{array} \right) \Bigg|_0^t$$

$$= \left(\begin{array}{c} 12t - 16 - 72t - 54 - 16e^{-3t} + 54e^{2t} \\ -24t + 32 - 36t - 27 + 32e^{-3t} + 27e^{2t} \end{array} \right)$$

$$= \left(\begin{array}{c} -60t - 70 + 16e^{-3t} + 54e^{2t} \\ -60t + 5 + 32e^{-3t} + 27e^{2t} \end{array} \right).$$