Math 2280 - Assignment 7

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Section 5.1 - 1, 7, 15, 21, 27 Section 5.2 - 1, 9, 15, 21, 39 Section 5.4 - 1, 8, 15, 25, 33

Section 5.1 - Matrices and Linear Systems

5.1.1 - Let

$$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix}.$$

Find

(a) 2A + 3B;
(b) 3A - 2B;
(c) AB;
(d) BA.

Solution -

(a)
$$2\mathbf{A} + 3\mathbf{B} = \begin{pmatrix} 4 & -6 \\ 8 & 14 \end{pmatrix} + \begin{pmatrix} 9 & -12 \\ 15 & 3 \end{pmatrix} = \begin{pmatrix} 13 & -18 \\ 23 & 17 \end{pmatrix}.$$

(b) $3\mathbf{A} - 2\mathbf{B} = \begin{pmatrix} 6 & -9 \\ 12 & 21 \end{pmatrix} - \begin{pmatrix} 6 & -8 \\ 10 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 19 \end{pmatrix}.$
(c) $\mathbf{AB} = \begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -9 & -11 \\ 47 & -9 \end{pmatrix}.$
(d) $\mathbf{BA} = \begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} -10 & -37 \\ 14 & -8 \end{pmatrix} \neq \begin{pmatrix} -9 & -11 \\ 47 & -9 \end{pmatrix}.$

5.1.7 - For the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix},$$

Calculate **AB**, and then compute the determinants of the matrices **A** and **B** above. Are your results consistent with the theorem to the effect that

$$det(\mathbf{AB}) = det(\mathbf{A})det(\mathbf{B})$$

for any two square matrices **A** and **B** of the same order?

Solution - We have:

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The determinants of **A** and **B** are:

$$det(\mathbf{A}) = \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 1 \times 4 - (-2) \times (-2) = 0,$$
$$det(\mathbf{B}) = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 2 \times 2 - 4 \times 1 = 0.$$

So, $det(\mathbf{AB}) = 0 = 0 \times 0 = det(\mathbf{A})det(\mathbf{B})$. Yes, it's consistent. You can sleep at night.

5.1.15 - Write the system below in the form $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$.

Solution -

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

5.1.21 For the system below, first verify that the given vectors are solutions of the system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system.

$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \mathbf{x};$$
$$\mathbf{x}_1 = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}.$$

Solution - If we plug x_1 into the system of equations we get:

$$\mathbf{x}_{1}^{\prime} = \begin{pmatrix} 2e^{t} \\ -3e^{t} \end{pmatrix},$$
$$\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} 2e^{t} \\ -3e^{t} \end{pmatrix} = \begin{pmatrix} 2e^{t} \\ -3e^{t} \end{pmatrix}.$$

So, \mathbf{x}_1 checks out. As for \mathbf{x}_2 we have:

$$\mathbf{x}_{2}' = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix},$$
$$\begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -2e^{2t} \end{pmatrix}.$$

So, \mathbf{x}_2 checks out, too. To show they're linearly independent we calculate the Wronskian:

$$W(\mathbf{x}_1, \mathbf{x}_2) = \begin{vmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{vmatrix} = -2e^{3t} - (-3e^{3t}) = e^{3t} \neq 0.$$

So, the solution vectors are linearly independent, and our general solution can be written as:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

5.1.27 For the system below, first verify that the given vectors are solutions of the system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system.

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x};$$
$$\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Solution - If we plug x_1 into the system of equations we get:

$$\mathbf{x}_{1}' = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{pmatrix}.$$

So, \mathbf{x}_1 checks out. As for \mathbf{x}_2 we have:

$$\mathbf{x}_{2}^{\prime} = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 0 \\ e^{-t} \end{pmatrix}.$$

So, \mathbf{x}_2 checks out, too. Finally, for \mathbf{x}_3 we have:

$$\mathbf{x}_{3}^{\prime} = \begin{pmatrix} 0\\ -e^{-t}\\ e^{-t} \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0\\ e^{-t}\\ -e^{-t} \end{pmatrix} = \begin{pmatrix} 0\\ -e^{-t}\\ e^{-t} \end{pmatrix}.$$

So, x_3 checks out as well. We have three solution vectors, and for a complete solution these vectors must be linearly independent, which we can check using the Wronskian:

$$W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{vmatrix} e^{2t} & e^{-t} & 0\\ e^{2t} & 0 & e^{-t}\\ e^{2t} & -e^{-t} & -e^{-t} \end{vmatrix} = 0 + 1 + 0 - (-1) - (-1) - 0 = 3 \neq 0.$$

So, the three solution vectors are linearly independent, and therefore our general solution can be written as:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1\\1\\1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1\\0\\-1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0\\1\\-1 \end{pmatrix} e^{-t}.$$

The Eigenvalue Method for Homogeneous Systems

5.2.1 - Apply the eigenvalue method to find the general solution to the system below. Use a computer or graphing calculator to construct a direction field and typical solution curves for the system.

$$\begin{array}{rcrcrcrc} x_1' &=& x_1 &+& 2x_2 \\ x_2' &=& 2x_1 &+& x_2 \end{array}$$

Solution - The vector-matrix form of the above first-order system is:

$$\mathbf{x}' = \left(\begin{array}{cc} 1 & 2\\ 2 & 1 \end{array}\right) \mathbf{x}$$

The characteristic polynomial of this matrix is:

$$\begin{vmatrix} 1-\lambda & 2\\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$$

So, the eigenvalues are $\lambda = 3, -1$. The associated eigenvectors are: For $\lambda = 3$:

$$\begin{pmatrix} -2 & 2\\ 2 & -2 \end{pmatrix} \begin{pmatrix} a\\ b \end{pmatrix} = \begin{pmatrix} 0\\ 0 \end{pmatrix},$$

and we see $\begin{pmatrix} a\\ b \end{pmatrix} = \begin{pmatrix} 1\\ 1 \end{pmatrix}$ works.

For $\lambda = -1$:

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and we see $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works.

So, our general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1\\1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1\\-1 \end{pmatrix} e^{-t}.$$

More room for Problem 5.2.1, if you need it.

Pirection Field



5.2.9 - Apply the eigenvalue method to find the particular solution to the initial value problem below. Use a computer or graphing calculator to construct a direction field and typical solution curves for the system.

$$\begin{array}{rcl}
x_1' &=& 2x_1 &-& 5x_2 \\
x_2' &=& 4x_1 &-& 2x_2 \\
x_1(0) &= 2, & x_2(0) &= 3.
\end{array}$$

Solution - The vector-matrix form of the above system of equations is:

$$\mathbf{x}' = \left(\begin{array}{cc} 2 & -5\\ 4 & -2 \end{array}\right) \mathbf{x}.$$

The characteristic polynomial for the matrix is:

$$\begin{vmatrix} 2-\lambda & -5\\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 + 16.$$

The roots of this polynomial are $\lambda = \pm 4i$, and so these are the eigenvalues. The associated eigenvectors will be:

$$\begin{pmatrix} 2-4i & -5\\ 4 & -2-4i \end{pmatrix} \begin{pmatrix} 5\\ 2-4i \end{pmatrix} = \begin{pmatrix} 0\\ 0 \end{pmatrix},$$

so $\begin{pmatrix} 5\\ 2-4i \end{pmatrix}$ for $\lambda = 4i$. For $\lambda = -4i$ the vector $\begin{pmatrix} 5\\ 2+4i \end{pmatrix}$ is an eigenvector.

Using these eigenvectors and eigenvalues along with equation 5.2.22 from the textbook we get the general solution:

$$\mathbf{x}(t) = c_1 \left[\begin{pmatrix} 5\\2 \end{pmatrix} \cos(4t) + \begin{pmatrix} 0\\4 \end{pmatrix} \sin(4t) \right] + c_2 \left[-\begin{pmatrix} 0\\4 \end{pmatrix} \cos(4t) + \begin{pmatrix} 5\\2 \end{pmatrix} \sin(4t) \right].$$

If we plug in $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ we get the system:

$$\begin{array}{rcl} 2 & = & 5c_1 \\ 3 & = & 2c_1 & - & 4c_2 \end{array}.$$

Solving this for c_1 and c_2 gives us $c_1 = \frac{2}{5}$, $c_2 = -\frac{11}{20}$. Plugging these values in we get the solution to our initial value problem:

$$\mathbf{x}(t) = \begin{pmatrix} 2\\3 \end{pmatrix} \cos\left(4t\right) + \begin{pmatrix} -\frac{11}{4}\\\frac{1}{2} \end{pmatrix} \sin\left(4t\right).$$

More room for Problem 5.2.9, if you need it.

Direction Field



5.2.15 - Apply the eigenvalue method to find the general solution to the system below. Use a computer or graphing calculator to construct a direction field and typical solution curves for the system.

Solution - The vector-matrix form of the above system is:

$$\mathbf{x}' = \left(\begin{array}{cc} 7 & -5\\ 4 & 3 \end{array}\right) \mathbf{x}.$$

The characteristic polynomial for the matrix is:

$$\begin{vmatrix} 7-\lambda & -5\\ 4 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) + 20 = \lambda^2 - 10\lambda + 41.$$

We can find the roots of this polynomial using the quadratic equation:

$$\lambda = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(41)}}{2} = 5 \pm 4i.$$

So, the eigenvalues are $5 \pm 4i$. For $\lambda = 5 + 4i$ a corresponding eigenvector is:

$$\left(\begin{array}{cc} 2-4i & -5\\ 4 & -2-4i \end{array}\right) \left(\begin{array}{c} 5\\ 2-4i \end{array}\right) = \left(\begin{array}{c} 0\\ 0 \end{array}\right).$$

Using equation 5.2.22 from the textbook we get the general solution:

$$\mathbf{x}(t) = c_1 e^{5t} \left[\begin{pmatrix} 5\\2 \end{pmatrix} \cos(4t) + \begin{pmatrix} 0\\4 \end{pmatrix} \sin(4t) \right] + c_2 e^{5t} \left[- \begin{pmatrix} 0\\4 \end{pmatrix} \cos(4t) + \begin{pmatrix} 5\\2 \end{pmatrix} \sin(4t) \right].$$

More room for Problem 5.2.15, if you need it.

Direction Field: Solution (urves 16

5.2.21 - The eigenvalues of the system below can be found by inspection and factoring. Apply the eigenvalue method to find a general solution to the system.

Solution - The vector-matrix format of the above system is:

$$\mathbf{x}' = \begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \mathbf{x}.$$

The characteristic polynomial for the matrix is:

$$\begin{vmatrix} 5-\lambda & 0 & -6\\ 2 & -1-\lambda & -2\\ 4 & -2 & -4-\lambda \end{vmatrix}$$

= $(5-\lambda)((-1-\lambda)(-4-\lambda)-4) + (-6)(-4-(-1-\lambda)4) = -\lambda(\lambda^2-1).$

So, the eigenvalues are $\lambda = 0, 1, -1$. Corresponding eigenvectors are: For $\lambda = 0$:

$$\begin{pmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = 1$:

$$\begin{pmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = -1$:

$$\begin{pmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So, the general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 6\\2\\5 \end{pmatrix} + c_2 \begin{pmatrix} 3\\1\\2 \end{pmatrix} e^t + c_3 \begin{pmatrix} 2\\1\\2 \end{pmatrix} e^{-t}.$$

5.2.39 For the matrix given below the zeros of the matrix make its characteristic polynomial easy to calculate. Find the general solution of $\mathbf{x}' = A\mathbf{x}$.

$$A = \left(\begin{array}{rrrr} -2 & 0 & 0 & 9\\ 4 & 2 & 0 & -10\\ 0 & 0 & -1 & 8\\ 0 & 0 & 0 & 1 \end{array}\right).$$

Solution - The characteristic polynomial for the matrix *A* above is:

$$\begin{vmatrix} -2-\lambda & 0 & 0 & 9\\ 4 & 2-\lambda & 0 & -10\\ 0 & 0 & -1-\lambda & 8\\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)(2-\lambda)(-2-\lambda).$$

So, there are 4 distinct eigenvalues, $\lambda = 1, -1, 2, -2$. Corresponding eigenvectors are:

For $\lambda = 1$:

$$\begin{pmatrix} -3 & 0 & 0 & 9\\ 4 & 1 & 0 & -10\\ 0 & 0 & -2 & 8\\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3\\ -2\\ 4\\ 1 \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0 \end{pmatrix}.$$

For $\lambda = -1$:

$$\left(\begin{array}{rrrr} -1 & 0 & 0 & 9\\ 4 & 3 & 0 & -10\\ 0 & 0 & 0 & 8\\ 0 & 0 & 0 & 2 \end{array}\right) \left(\begin{array}{r} 0\\ 0\\ 1\\ 0 \end{array}\right) = \left(\begin{array}{r} 0\\ 0\\ 0\\ 0 \end{array}\right).$$

For $\lambda = 2$:

$$\begin{pmatrix} -4 & 0 & 0 & 9 \\ 4 & 0 & 0 & -10 \\ 0 & 0 & -3 & 8 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = -2$:

$$\begin{pmatrix} 0 & 0 & 0 & 9 \\ 4 & 4 & 0 & -10 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So, the general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ -2 \\ 4 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} e^{-2t}.$$

Section 5.4 - Multiple Eigenvalue Solutions

5.4.1 - Find a general solution to the system of differential equations below.

$$\mathbf{x}' = \begin{pmatrix} -2 & 1\\ -1 & -4 \end{pmatrix} \mathbf{x}$$

Solution - The matrix for this system has eigenvalues:

$$\begin{vmatrix} -2 - \lambda & 1 \\ -1 & -4 - \lambda \end{vmatrix} = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2.$$

We have eigenvalues $\lambda = -3, -3$. So, there's only one eigenvalue, and it has multiplicity 2. The eigenvectors for this eigenvalue must satisfy:

$$\left(\begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array}\right)\mathbf{v}_1 = \left(\begin{array}{c} 0 \\ 0 \end{array}\right).$$

 $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works, and there is no other linearly independent eigenvector.

We therefore need a length 2 chain of generalized eigenvectors. First, we calculate:

$$\left(\begin{array}{rrr}1&1\\-1&-1\end{array}\right)^2 = \left(\begin{array}{rrr}0&0\\0&0\end{array}\right).$$

So, $\mathbf{v}_2 = \left(egin{array}{c} 1 \\ 0 \end{array}
ight)$ works with

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathbf{v}_1.$$

The general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{-3t}.$$

5.4.8 Find a general solution to the system of differential equations below.

$$\mathbf{x}' = \begin{pmatrix} 25 & 12 & 0\\ -18 & -5 & 0\\ 6 & 6 & 13 \end{pmatrix} \mathbf{x}$$

Solution - The eigenvalues for the coefficient matrix are:

$$\begin{vmatrix} 25 - \lambda & 12 & 0 \\ -18 & -5 - \lambda & 0 \\ 6 & 6 & 13 - \lambda \end{vmatrix} = (13 - \lambda)[(25 - \lambda)(-5 - \lambda) - (12)(-18)]$$
$$= (13 - \lambda)(\lambda^2 - 20\lambda + 91) = -(\lambda - 13)^2(\lambda - 7).$$

So, the eigenvalues are $\lambda = 7, 13, 13$.

For $\lambda = 7$ the eigenvector must satisfy:

$$\left(\begin{array}{rrrr} 18 & 12 & 0\\ -18 & -12 & 0\\ 6 & 6 & 6 \end{array}\right) \mathbf{v} = \left(\begin{array}{r} 0\\ 0\\ 0 \end{array}\right).$$

The vector $\mathbf{v} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ works.

For $\lambda = 13$ an eigenvector must satisfy:

$$\left(\begin{array}{rrrr} 12 & 12 & 0\\ -18 & -18 & 0\\ 6 & 6 & 0 \end{array}\right) \mathbf{v} = \left(\begin{array}{r} 0\\ 0\\ 0 \end{array}\right).$$

The linearly independent vectors $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ work.

The general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2\\ -3\\ 1 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} 1\\ -1\\ 0 \end{pmatrix} e^{13t} + c_3 \begin{pmatrix} 0\\ 0\\ 1 \end{pmatrix} e^{13t}.$$

5.4.15 - Find a general solution to the system of differential equations below.

$$\mathbf{x}' = \begin{pmatrix} -2 & -9 & 0\\ 1 & 4 & 0\\ 1 & 3 & 1 \end{pmatrix} \mathbf{x}$$

Solution - The eigenvalues of the coefficient matrix are:

$$\begin{vmatrix} -2 - \lambda & -9 & 0\\ 1 & 4 - \lambda & 0\\ 1 & 3 & 1 - \lambda \end{vmatrix} = (1 - \lambda)[(-2 - \lambda)(4 - \lambda) - 1(-9)]$$
$$= (1 - \lambda)(\lambda^2 - 2\lambda + 1)^2 = -(\lambda - 1)^3.$$

The eigenvalues are $\lambda = 1, 1, 1$. There is only one eigenvalue, and it has multiplicity 3. The eigenvectors for this eigenvalue must satisfy:

$$\left(\begin{array}{rrr} -3 & -9 & 0\\ 1 & 3 & 0\\ 1 & 3 & 0 \end{array}\right) \mathbf{v} = \left(\begin{array}{r} 0\\ 0\\ 0 \end{array}\right).$$

The linearly independent vectors $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ both work.

So, we need a single length 2 chain. To find this, we calculate the matrix product:

$$\begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The vector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is a candidate for the top vector in our chain, and the corresponding base vector is:

$$\begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \mathbf{v}_1.$$

This is a length 2 chain. For the additional length 1 chain we need an eigenvector that is independent of \mathbf{v}_1 above, and using our earlier

results we see $\begin{pmatrix} 0\\0\\1 \end{pmatrix}$ works.

The general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 0\\0\\1 \end{pmatrix} e^t + c_2 \begin{pmatrix} -3\\1\\1 \end{pmatrix} e^t + c_3 \left(\begin{pmatrix} -3\\1\\1 \end{pmatrix} t + \begin{pmatrix} 1\\0\\0 \end{pmatrix} \right) e^t.$$

5.4.25 - Find a general solution to the system of differential equations below. The eigenvalues of the matrix are given.

$$\mathbf{x}' = \begin{pmatrix} -2 & 17 & 4 \\ -1 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix} \mathbf{x}; \quad \lambda = 2, 2, 2.$$

Solution - An eigenvector for the system must satisfy:

$$\left(\begin{array}{rrr} -4 & 17 & 4\\ -1 & 4 & 1\\ 0 & 1 & 0 \end{array}\right) \mathbf{v} = \left(\begin{array}{r} 0\\ 0\\ 0 \end{array}\right).$$

The vector $\begin{pmatrix} 1\\0\\1 \end{pmatrix}$ is the only linearly independent eigenvector that works. So, we need a length 3 (yikes!) generalized eigenvector. To find it we calculate:

$$\begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 4 & 1 \\ 0 & 0 & 0 \\ -1 & 4 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A good test vector for the top vector of our chain is $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. With this vector we get:

$$\begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} = \mathbf{v}_2,$$
$$\begin{pmatrix} -4 & 17 & 4 \\ -1 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{v}_1.$$

So, the test vector works, and we have a length 3 chain of generalized eigenvectors. The corresponding general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} e^{2t} + c_2 \left(\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} \right) e^{2t} + c_3 \left(\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) e^{2t}.$$

5.4.33 - The characteristic equation of the coefficient matrix *A* of the system

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{pmatrix} \mathbf{x}$$

is

$$\phi(\lambda) = (\lambda^2 - 6\lambda + 25)^2 = 0.$$

Therefore, *A* has the repeated complex conjugate pair $3 \pm 4i$ of eigenvalues. First show that the complex vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1\\i\\0\\0 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 0\\0\\1\\i \end{pmatrix}^1$$

form a length 2 chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ associated with the eigenvalue $\lambda = 3 - 4i$. Then calculate the real and imaginary parts of the complex-valued solutions

$$\mathbf{v}_1 e^{\lambda t}$$
 and $(\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t}$

to find four independent real-valued solutions of $\mathbf{x}' = A\mathbf{x}$.

Solution - First, we verify that the two vectors are a chain:

¹Note in the textbook there's a typo in this vector.

$$(A - \lambda I) = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix},$$
$$(A - \lambda I)\mathbf{v}_{2} = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} = \mathbf{v}_{1},$$
$$(A - \lambda I)\mathbf{v}_{1} = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix} \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So, it's a chain. This means we have solutions:

$$\mathbf{x}_{1} = \begin{pmatrix} 1\\i\\0\\0 \end{pmatrix} e^{(3-4i)t},$$
$$\mathbf{x}_{2} = \begin{pmatrix} \begin{pmatrix} 1\\i\\0\\0 \end{pmatrix} t + \begin{pmatrix} 0\\0\\1\\i \end{pmatrix} \end{pmatrix} e^{(3-4i)t}.$$

Breaking these up into real and imaginary parts we get:

$$e^{(3-4i)t} = e^{3t}(\cos(4t) - i\sin(4t)),$$

$$Re(\mathbf{x}_1) = \begin{pmatrix} \cos(4t) \\ \sin(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} \qquad Im(\mathbf{x}_1) = \begin{pmatrix} -\sin(4t) \\ \cos(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t},$$

$$Re(\mathbf{x}_2) = \begin{pmatrix} t\cos(4t) \\ t\sin(4t) \\ \cos(4t) \\ \sin(4t) \end{pmatrix} e^{3t} \qquad Im(\mathbf{x}_2) = \begin{pmatrix} -t\sin(4t) \\ -t\cos(4t) \\ -\sin(4t) \\ \cos(4t) \end{pmatrix} e^{3t}.$$

So, our final solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos(4t) \\ \sin(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -\sin(4t) \\ \cos(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} t\cos(4t) \\ t\sin(4t) \\ \cos(4t) \\ \sin(4t) \end{pmatrix} e^{3t} + c_4 \begin{pmatrix} -t\sin(4t) \\ t\cos(4t) \\ -\sin(4t) \\ \cos(4t) \end{pmatrix} e^{3t}.$$