Math 2280 - Assignment 5

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Section 3.4 - 1, 5, 18, 21

Section 3.5 - 1, 11, 23, 28, 35, 47, 56

Section 3.6 - 1, 2, 9, 17, 24

Section 3.7 - 1, 5, 10, 17, 19

Section 3.4 - Mechanical Vibrations

3.4.1 - Determine the period and frequency of the simple harmonic motion of a 4-kg mass on the end of a spring with spring constant $16N/m$.

Solution - The differential equation $4x'' + 16x = 0$ can be rewritten as $x'' + 4x = 0$. This gives us $\omega^2 = 4$, and so the angular frequency is $\omega = 2$. From this we get the frequency and the period are:

$$f = \frac{2}{2\pi} = \frac{1}{\pi},$$

$$T = \frac{1}{f} = \frac{1}{\frac{1}{\pi}} = \pi.$$

3.4.5 - Assume that the differential equation of a simple pendulum of length L is $L\theta'' + g\theta = 0$, where $g = GM/R^2$ is the gravitational acceleration at the location of the pendulum (at distance R from the center of the earth; M denotes the mass of the earth).

Two pendulums are of lengths L_1 and L_2 and - when located at the respective distances R_1 and R_2 from the center of the earth - have periods p_1 and p_2. Show that

$$\frac{p_1}{p_2} = \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}.$$

Solution - The differential equation governing the motion of our pendulum is:

$$\theta'' + \frac{g}{L}\theta = 0.$$

From this we get $\omega = \sqrt{\frac{g}{L}}$, and so the period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$.

Ther period of pendulum 1 will therefore be:

$$p_1 = 2\pi\sqrt{\frac{L_1}{\frac{GM}{R_1^2}}} = 2\pi\sqrt{\frac{L_1R_1^2}{GM}}.$$

Similarly, the period of pendulum 2 will be:

$$p_2 = 2\pi\sqrt{\frac{L_2R_2^2}{GM}}.$$

So, the ratio is:

$$\frac{p_1}{p_2} = \frac{2\pi\frac{R_1\sqrt{L_1}}{\sqrt{GM}}}{2\pi\frac{R_2\sqrt{L_2}}{\sqrt{GM}}} = \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}.$$

3.4.18 - A mass m is attached to both a spring (with spring constant k) and a dashpot (with dampring constant c). The mass is set in motion with initial position x_0 and initial velocity v_0. Find the position function $x(t)$ and determine whether the motion is overdamped, critically damped, or underdamped. If it is underdamped, write the position function in the form $x(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$. Also, find the undamped position function $u(t) = C_0\cos(\omega_0 t - \alpha_0)$ that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected (so $c = 0$). Finally, construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$.

$$m = 2, \quad c = 12, \quad k = 50,$$

$$x_0 = 0, \quad v_0 = -8.$$

Solution - The discriminant of the characteristic polynomial is:

$$c^2 - 4mk = 12^2 - 4(2)(50) = 144 - 400 = -256 < 0.$$

So, the motion will be underdamped. The roots of the characteristic polynomial:

$$2r^2 + 12r + 50$$

are (using the quadratic equation):

$$r = \frac{-12 \pm \sqrt{-256}}{2(2)} = -3 \pm 4i.$$

So, the solution is:

$$x(t) = c_1 e^{-3t} \cos(4t) + c_2 e^{-3t} \sin(4t).$$

Plugging in the initial conditions we have:

$$x(0) = c_1 = 0,$$

and therefore

$$x'(t) = -3c_2 e^{-3t} \sin(4t) + 4c_2 e^{-3t} \cos(4t)$$
$$\Rightarrow x'(0) = 4c_2 = -8 \Rightarrow c_2 = -2.$$

So,

$$x(t) = -2e^{-3t} \sin(4t),$$

which we can write as:

$$x(t) = -2e^{-3t} \cos\left(4t - \frac{\pi}{2}\right).$$

Now, without damping we would get

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{50}{2}} = 5,$$

and our solution would be:

$$u(t) = c_1 \cos(5t) + c_2 \sin(5t).$$

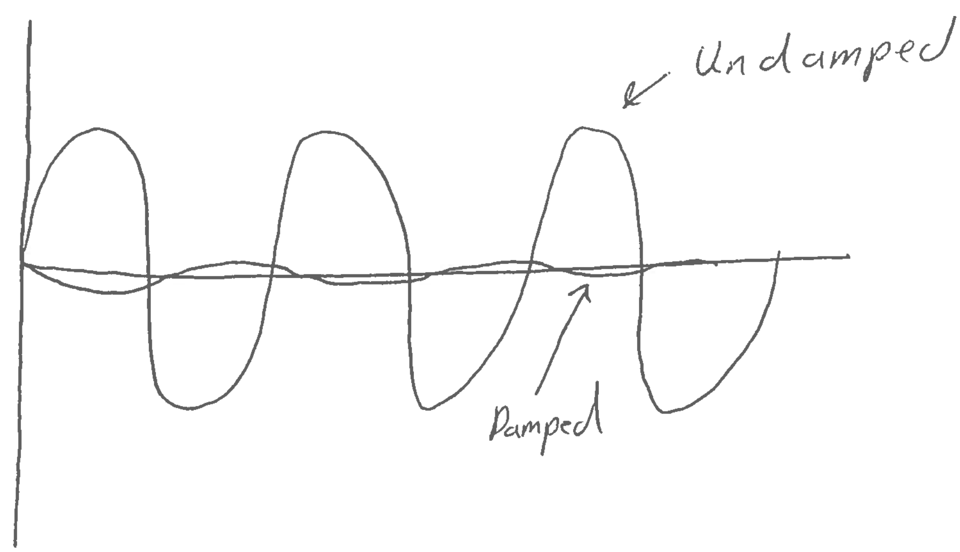
If we plug in $u(0) = c_1 = 0$ we get:

$$u'(t) = 5c_2 \cos(4t),$$

and so $u'(0) = 5c_2 = -8 \Rightarrow c_2 = -\frac{8}{5}$. Therefore, the undamped motion would be:

$$u(t) = -\frac{8}{5}\sin(5t) = -\frac{8}{5}\cos\left(5t - \frac{\pi}{2}\right).$$

The graphs of these two functions (damped and undamped) look like:



3.4.21 - Same as problem 3.4.18, except with the following values:

$$m = 1, \quad c = 10, \quad k = 125,$$

$$x_0 = 6, \quad v_0 = 50.$$

Solution - The characteristic equation for this system will be:

$$r^2 + 10r + 25$$

which has roots:

$$r = \frac{-10 \pm \sqrt{10^2 - 4(1)(125)}}{2} = -5 \pm 10i.$$

So, our solution is:

$$x(t) = e^{-5t}(c_1 \cos(10t) + c_2 \sin(10t)),$$

with

$$x'(t) = 10e^{-5t}(-c_1 \sin(10t) + c_2 \cos(10t)) - 5e^{-5t}(c_1 \cos(10t) + c_2 \sin(10t)).$$

If we plug in our initial conditions we get:

$$x(0) = c_1 = 6,$$

and

$$x'(0) = 10c_2 - 5c_1 = 10c_2 - 30 = 50 \Rightarrow c_2 = 8.$$

So, our solution is:

$$x(t) = e^{-5t}(6\cos(10t) + 8\sin(10t)).$$

We can rewrite this as:

$$10e^{-5t}\left(\frac{3}{5}\cos(10t)+\frac{4}{5}\sin(10t)\right)=10e^{-5t}\cos(10t-\alpha),$$

where $\alpha=\tan^{-1}\left(\frac{4}{3}\right)$.

As for the undamped case we have:

$$\omega=\sqrt{\frac{125}{1}}=5\sqrt{5}.$$

So, our solution is:

$$u(t)=c_1\cos(5\sqrt{5}t)+c_2\sin(5\sqrt{5}t),$$
$$u'(t)=-5\sqrt{5}c_1\sin(5\sqrt{5}t)+5\sqrt{5}c_2\cos(5\sqrt{5}t).$$

If we plug in the initial conditions we get:

$$u(0)=c_1=6,$$
$$u'(0)=5\sqrt{5}c_2=50\Rightarrow c_2=2\sqrt{5}.$$

So, our solution is:

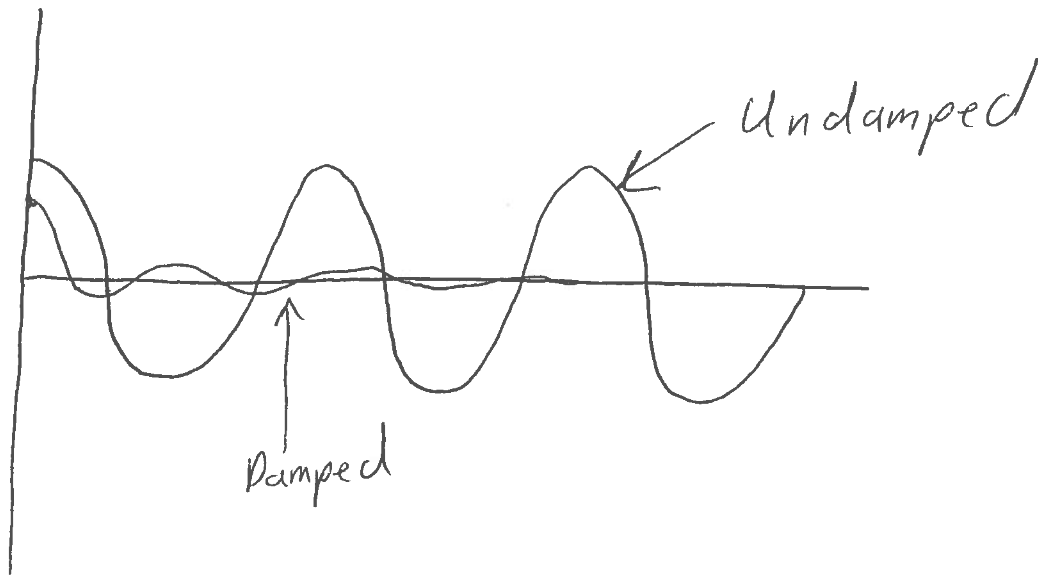
$$u(t)=6\cos(5\sqrt{5}t)+2\sqrt{5}\sin(5\sqrt{5}t).$$

Writing this as just a cosine function we get that the amplitude is:

$$C=\sqrt{6^2+(2\sqrt{5})^2}=\sqrt{36+20}=\sqrt{56}=2\sqrt{14},$$
$$\alpha=\tan^{-1}\left(\frac{2\sqrt{5}}{6}\right).$$

So, $u(t) = 2\sqrt{14}\cos\left(5\sqrt{5}t - \tan^{-1}\left(\frac{\sqrt{5}}{3}\right)\right)$.

The graphs of the damped and undamped solutions, $x(t)$ and $u(t)$, respectively, are below:



Section 3.5 - Nonhomogeneous Equations and Undetermined Coefficients

3.5.1 - Find a particular solution, y_p, to the differential equation

$$y'' + 16y = e^{3x}.$$

Solution - We guess the solution will be of the form:

$$y_p = Ae^{3x},$$

$$y_p' = 3Ae^{3x},$$

$$y_p'' = 9Ae^{3x}.$$

Plugging these into the ODE we get:

$$y_p'' + 16y_p = 25Ae^{3x} = e^{3x}.$$

So, $A = \frac{1}{25}$, and we get:

$$y_p = \frac{1}{25}e^{3x}.$$

3.5.11 - Find a particular solution, y_p, to the differential equation

$$y^{(3)} + 4y' = 3x - 1.$$

Solution - The corresponding homogeneous equation is:

$$y^{(3)} + 4y' = 0,$$

which has characteristic polynomial:

$$r^3 + 4r = r(r^2 + 4).$$

The roots of this polynomial are $r = 0, \pm 2i$, and the corresponding homogeneous solution is:

$$y_h = c_1 + c_2 \sin(2x) + c_3 \cos(2x).$$

Now, our initial "guess" for the form of the particular solution would be:

$$y_p = Ax + B.$$

However, the two terms here are not linearly independent of the homogeneous solution, and so we need to multiply our guess by x to get:

$$y_p = Ax^2 + Bx.$$

From here we get:

$$y_p' = 2Ax + B,$$

$$y_p'' = 2A,$$

$$y_p^{(3)} = 0.$$

Plugging these into our ODE we get:

$$0 + 8Ax + 4B = 3x - 1.$$

From this we get $A = \frac{3}{8}$, and $B = -\frac{1}{4}$. So, our particular solution is:

$$y_p = \frac{3}{8}x^2 - \frac{1}{4}x.$$

3.5.23 - Set up the appropriate form of a particular solution y_p, but do not determine the values of the coefficients.[1]

$$y'' + 4y = 3x\cos 2x.$$

Solution - The corresponding homogeneous equation is:

$$y'' + 4y = 0.$$

The characteristic polynomial is:

$$r^2 + 4,$$

which has roots $\pm 2i$. So, the form of the homogeneous solution is:

$$y(x) = c_1 \sin(2x) + c_2 \cos(2x).$$

Now, the guess for our particular solution would be:

$$y_p(x) = (Ax + B)\sin(2x) + (Cx + D)\cos(2x).$$

This guess is, however, not independent of our homogeneous solution, and so we must multiply it by x to get:

$$y_p = (Ax^2 + Bx)\sin(2x) + (Cx^2 + Dx)\cos(2x).$$

[1]Unless you really, really want to.

3.5.28 - Same instructions as Problem 3.5.23, but with the differential equation

$$y^{(4)} + 9y'' = (x^2 + 1)\sin 3x.$$

Solution - The corresponding homogeneous equation is:

$$y^{(4)} + 9y'' = 0.$$

The characteristic polynomial for this differential equation is:

$$r^4 + 9r^2 = r^2(r^2 + 9).$$

This polynomial has roots $r = 0, 0, \pm 3i$. So, the homogeneous solution has the form:

$$y_h = c_1 + c_2 x + c_3 \sin(3x) + c_4 \cos(3x).$$

Our initial "guess" for the particular solution would be:

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x).$$

However, the terms here would not be independent of the homogeneous solution, and so we must multiply our guess by x to get:

$$y_p = (Ax^3 + Bx^2 + Cx)\sin(3x) + (Dx^3 + Ex^2 + Fx)\cos(3x).$$

3.5.35 - Solve the initial value problem

$$y'' - 2y' + 2y = x + 1;$$
$$y(0) = 3, \quad y'(0) = 0.$$

Solution - The corresponding homogeneous equation is:

$$y'' - 2y' + 2y = 0.$$

The characteristic polynomial is:

$$r^2 - 2r + 2.$$

This quadratic has roots $r = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2} = 1 \pm i$. So, the general form of the homogeneous solution is:

$$y_h = c_1 e^x \cos(x) + c_2 e^x \sin(x).$$

Our "guess" for the particular solution will be:

$$y_p = Ax + B,$$

with

$$y_p' = A,$$
$$y_p'' = 0.$$

Plugging these into our differential equation we get:

$$0 - 2A + 2(Ax + B) = x + 1.$$

Equating coefficients we get $A = \frac{1}{2}, B = 1$. So,

$$y_p = \frac{1}{2}x + 1.$$

Therefore, the general form of our solution is:

$$y = y_h + y_p = c_1 e^x \cos(x) + c_2 e^x \sin(x) + \frac{1}{2}x + 1,$$

$$y' = (c_1 + c_2)e^x \cos(x) + (c_2 - c_1)e^x \sin(x) + \frac{1}{2}.$$

Plugging in our initial conditions we get:

$$y(0) = 3 = c_1 + 1,$$

$$y'(0) = 0 = c_1 + c_2 + \frac{1}{2}.$$

Solving this system we get $c_1 = 2, c_2 = -\frac{5}{2}$. So, our solution is:

$$y(x) = 2e^x \cos(x) - \frac{5}{2}e^x \sin(x) + \frac{1}{2}x + 1.$$

3.5.47 - Use the method of variation of parameters to find a particular solution to the differential equation

$$y'' + 3y' + 2y = 4e^x.$$

Solution - The corresponding homogeneous equation is:

$$y'' + 3y' + 2y = 0.$$

This homogeneous equation has characteristic polynomial:

$$r^2 + 3r + 2 = (r+2)(r+1).$$

The roots of this polynomial are $r = -1, -2$, and so the general form of the homogeneous solution is:

$$y_h = c_1 e^{-2x} + c_2 e^{-x}.$$

The corresponding Wronskian is:

$$W(e^{-2x}, e^{-x}) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = e^{-3x}.$$

Using the method of variation of parameters to get a particular solution we have:

$$y_p = -e^{-2x}\int \frac{e^{-x}(4e^x)}{e^{-3x}}dx + e^{-x}\int \frac{e^{-2x}(4e^x)}{e^{-3x}}dx$$

$$= -4e^{-2x}\int e^{3x}dx + 4e^{-x}\int e^{2x}dx = -\frac{4}{3}e^x + 2e^x = \frac{2}{3}e^x.$$

We can quickly check this:

$$\frac{2}{3}e^x + 3\left(\frac{2}{3}e^x\right) + 2\left(\frac{2}{3}e^x\right) = 4e^x.$$

3.5.56 - Same instructions as Problem 3.5.47, but with the differential equation

$$y'' - 4y = xe^x.$$

Solution - The corresponding homogeneous equation is:

$$y'' - 4y = 0.$$

The characteristic polynomial is:

$$r^2 - 4 = (r-2)(r+2).$$

The roots of this polynomial are $r = \pm 2$, and so the general form of the homogeneous solution is:

$$y_h = c_1 e^{2x} + c_2 e^{-2x}.$$

The corresponding Wronskian is:

$$W(e^{2x}, e^{-2x}) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4.$$

Using the method of variation of parameters to get a particular solution we have:

$$
\begin{aligned}
y_p &= -e^{2x} \int \frac{e^{-2x}(xe^x)}{-4} dx + e^{-2x} \int \frac{e^{2x}(xe^x)}{-4} dx \\
&= \frac{1}{4} \left(e^{2x} \int xe^{-x} dx - e^{-2x} \int xe^{3x} dx \right) \\
&= \frac{1}{4} \left(e^{2x} \left(-xe^{-x} - e^{-x} \right) - e^{-2x} \left(\frac{1}{3} xe^{3x} - \frac{1}{9} e^{3x} \right) \right) \\
&= \frac{1}{4} \left(-xe^x - e^x - \frac{1}{3} xe^x + \frac{1}{9} e^x \right) \\
&= -\frac{1}{3} xe^x - \frac{2}{9} e^x.
\end{aligned}
$$

Section 3.6 - Forced Oscillations and Resonance

3.6.1 - Express the solution of the initial value problem

$$x'' + 9x = 10\cos 2t;$$

$$x(0) = x'(0) = 0,$$

as a sum of two oscillations in the form:

$$x(t) = C\cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t.$$

Solution - The corresponding homogeneous equation is:

$$x'' + 9x = 0,$$

which has characteristic polynomial:

$$r^2 + 9$$

The roots of this polynomial are $r = \pm 3i$, and so the general form of the homogeneous solution is:

$$x_h = c_1\cos(3t) + c_2\sin(3t).$$

As for the particular solution, we guess it's of the form:

$$x_p = A\cos(2t) + B\sin(2t).$$

The corresponding derivatives are:

$$x_p' = -2A\sin(2t) + 2B\cos(2t),$$
$$x_p'' = -4A\cos(2t) - 4B\sin(2t).$$

Plugging these into the ODE we get:

$$x_p'' + 9x_p = 5A\cos(2t) + 5B\sin(2t) = 10\cos(2t).$$

So,$A = 2, B = 0$, and

$$x_p = 2\cos(2t).$$

So, the general solution will be:

$$x(t) = c_1\cos(3t) + c_2\sin(3t) + 2\cos(2t),$$

with

$$x'(t) = -3c_1\sin(3t) + 3c_2\cos(3t) - 4\sin(2t).$$

Plugging in our initial conditions gives us:

$$x(0) = 0 = 2 + c_1$$
$$x'(0) = 0 = 3c_2.$$

So, $c_1 = -2, c_2 = 0$, and our solution is:

$$x(t) = -2\cos(3t) + 2\cos(2t).$$

This is already in the proper form.

3.6.2 - Same instructions as Problem 3.6.1, but with the initial value problem:

$$x'' + 4x = 5\sin 3t;$$

$$x(0) = x'(0) = 0.$$

Solution - The corresponding homogeneous equation is:

$$x'' + 4x = 0.$$

The characteristic polynomial for this equation is:

$$r^2 + 4 = 0,$$

which has roots $r = \pm 2i$. So, the general form of the homogeneous solution is:

$$x_h = c_1 \cos(2t) + c_2 \sin(2t).$$

As for the particular solution, we guess it's of the form:

$$x_p = A\cos(3t) + B\sin(3t),$$

with corresponding derivatives

$$x_p' = -3A\sin(3t) + 3B\cos(3t),$$

$$x_p'' = -9A\cos(3t) - 9B\sin(3t).$$

Plugging these into the ODE we get:

$$x_p'' + 4x_p = -5A\cos(3t) - 5B\sin(3t) = 5\sin(3t).$$

So, $A = 0, B = -1$, and our particular solution is:

$$x_p = -\sin(3t).$$

Our general solution is then:

$$x(t) = x_h + x_p = c_1\cos(2t) + c_2\sin(2t) - \sin(3t),$$

with corresponding derivative

$$x'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) - 3\cos(3t).$$

Plugging in our initial conditions we get:

$$x(0) = 0 = c_1$$

$$x'(0) = 0 = 2c_2 - 3.$$

So, $c_1 = 0, c_2 = \frac{3}{2}$, and our solution is:

$$x(t) = \frac{3}{2}\sin(2t) - \sin(3t).$$

Using the identity $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$ we can convert this to the desired form:

$$x(t) = \frac{3}{2}\cos\left(2t - \frac{\pi}{2}\right) - \cos\left(3t - \frac{\pi}{2}\right).$$

3.6.9 - Find the steady periodic solution $x_{sp}(t) = C\cos(\omega t - \alpha)$ of the given equation $mx'' + cx' + kx = F(t)$ with periodic forcing function $F(t)$ of frequency ω. Then graph $x_{sp}(t)$ together with (for comparison) the adjusted forcing function $F_1(t) = F(t)/m\omega$.

$$2x'' + 2x' + x = 3\sin 10t.$$

Solution - We first note the the corresponding homogeneous equation is:

$$2x'' + 2x' + x = 0,$$

which has characteristic polynomial

$$2r^2 + 2r + 1,$$

with roots $r = \dfrac{-2 \pm \sqrt{2^2 - 4(2)(1)}}{2(2)} = -\dfrac{1}{2} \pm \dfrac{1}{2}i.$

So, the general form of the homogeneous solution is:

$$x_h = c_1 e^{-\frac{1}{2}t}\cos\left(\frac{1}{2}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{1}{2}t\right).$$

The steady periodic solution is a particular solution, and our "guess" for the particular solution is:

$$x_{sp} = A\cos(10t) + B\sin(10t).$$

These terms are linearly independent of our homogeneous solution, so this is a good guess. Its corresponding derivatives are:

$$x'_{sp} = -10A\sin(10t) + 10B\cos(10t),$$

$$x''_{sp} = -100A\cos(10t) - 100B\sin(10t).$$

Plugging these into our ODE we get:

$$2x''_{sp} + 2x'_{sp} + x_{sp} =$$
$$(-200A+20B+A)\cos(10t)+(-200B-20A+B)\sin(10t) = 3\sin(10t).$$

From these we get the pair of linear equations:

$$-199A + 20B = 0,$$

$$-20A + 199B = 3.$$

The solution to this system is:

$$A = -\frac{60}{40001},$$

$$B = -\frac{597}{40001}.$$

So,

$$x_{sp} = -\frac{1}{40001}(60\cos(10t) + 597\sin(10t)).$$

To express this in the proper form we have:

$$C = \sqrt{\left(-\frac{60}{40001}\right)^2 + \left(-\frac{597}{40001}\right)^2} = \sqrt{\frac{360009}{40001^2}} = \frac{3}{\sqrt{40001}},$$

and

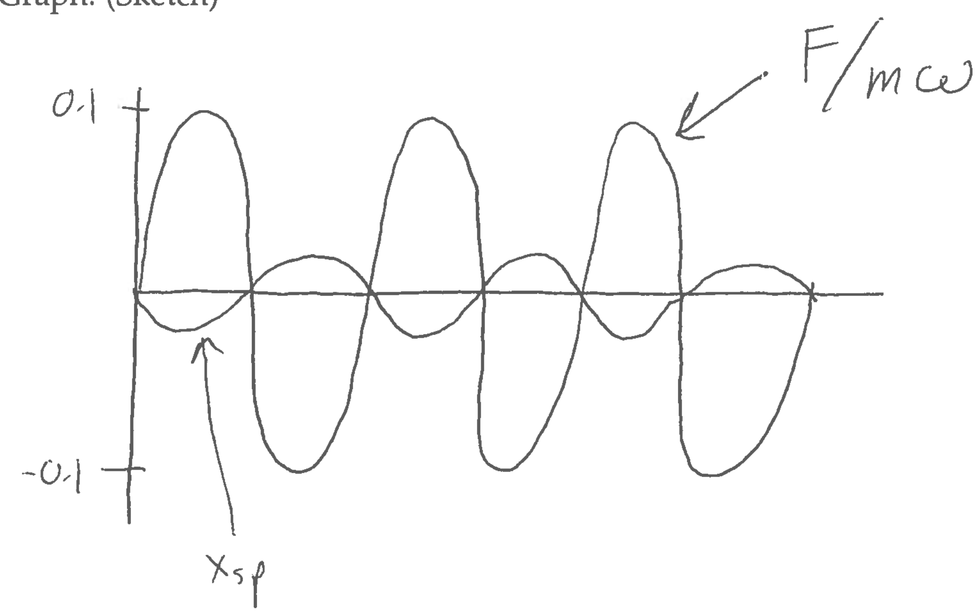
$$\alpha = \tan^{-1}\left(\frac{-597}{-60}\right) = \pi + \tan^{-1}\left(\frac{199}{20}\right).$$

So,

$$x_{sp} = C\cos\left(10t - \alpha\right),$$

with $C = \dfrac{3}{\sqrt{40001}}$ and $\alpha = \pi + \tan^{-1}\left(\frac{199}{20}\right)$.

Graph: (Sketch)



3.6.17 - Suppose we have a forced mass-spring-dashpot system with equation:

$$x'' + 6x' + 45x = 50\cos\omega t.$$

Investigate the possibility of practical resonance of this system. In particular, find the amplitude $C(\omega)$ of steady periodic forced oscillations with frequency ω. Sketch the graph of $C(\omega)$ and find the practical resonance frequency ω (if any).

Solution - Using the equation for $C(\omega)$ from section 3.6 of the textbook:

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}},$$

with $F_0 = 50, m = 1, c = 6$, and $k = 45$ we get:

$$C(\omega) = \frac{50}{\sqrt{(45 - \omega^2)^2 + (6\omega)^2}}.$$

We get practical resonance when $C(\omega)$ is maximized, which will be when the denominator is minimized. The denominator will be minimized when

$$(45 - \omega^2)^2 + (6\omega)^2 = w^4 - 54\omega^2 + 2025 = B(\omega)$$

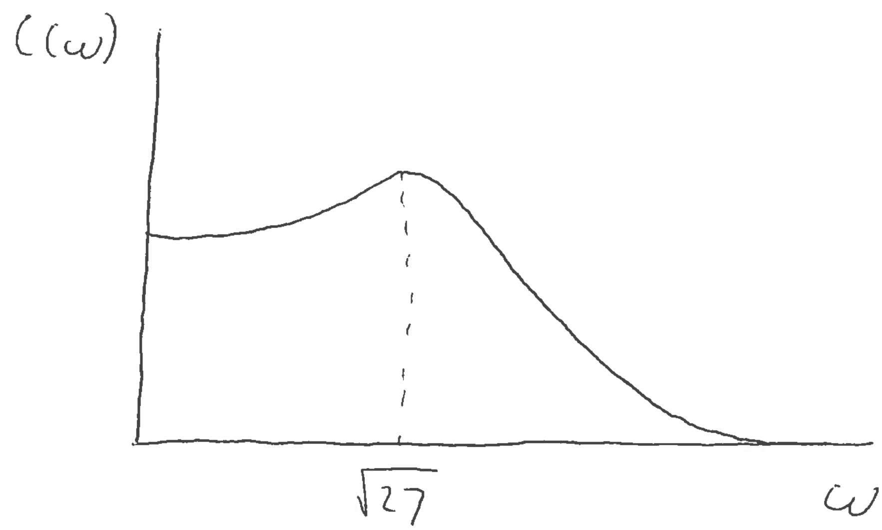
is minimized. Taking this functions first and second derivatives gives us:

$$B'(\omega) = 4\omega^3 - 108\omega = (4\omega^2 - 108)\omega,$$

$$B''(\omega) = 12\omega^2 - 108.$$

Now, $B'(\omega) = 0$ when $\omega = 0$ or $\omega = \pm\sqrt{27}$. Plugging these values into $B''(\omega)$ we find $B''(0) < 0$ and $B''(\pm\sqrt{27}) > 0$. So, $\omega = 0$ is a local max, while $\omega = \pm\sqrt{27}$ are minimums. As $\omega = -\sqrt{27}$ does not make physical sense, we see that practical resonance occurs at $\omega = \sqrt{27}$.

Graph of $C(\omega)$: (Sketch)



3.6.24 - A mass on a spring without damping is acted on by the external force $F(t) = F_0 \cos^3 \omega t$. Show that there are *two* values of ω for which resonance occurs, and find both.

Solution - Using the trigonometric identity

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

we have

$$F_0 \cos^3(\omega t) = \frac{F_0 \cos(\omega t) + F_0 \cos(\omega t)\cos(2\omega t)}{2}.$$

If we use the trigonometric identity

$$\cos(A)\cos(B) = \frac{\cos(A+B) + \cos(A-B)}{2}$$

we get:

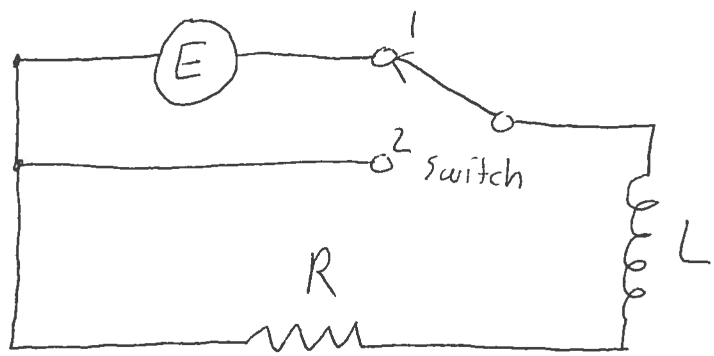
$$\frac{F_0 \cos(\omega t) + F_0 \cos(\omega t)\cos(2\omega t)}{2} = \frac{3F_0 \cos(\omega t)}{4} + \frac{F_0 \cos(3\omega t)}{4}.$$

So, resonance occurs when $\sqrt{\frac{k}{m}} = \omega$ *or* $\sqrt{\frac{k}{m}} = 3\omega$.

Section 3.7 - Electrical Circuits

3.7.1 This problem deals with the RL circuit pictured below. It is a series circuit containing an inductor with an inductance of L henries, a resistor with a resistance of R ohms, and a source of electromotive force (emf), but no capacitor. In this case the equation governing our system is the first-order equation

$$LI' + RI = E(t).$$



Suppose that $L = 5H$, $R = 25\Omega$, and the source E of emf is a battery supplying $100V$ to the circuit. Suppose also that the switch has been in position 1 for a long time, so that a steady current of $4A$ is flowing in the circuit. At time $t = 0$, the switch is thrown to position 2, so that $I(0) = 4$ and $E = 0$ for $t \geq 0$. Find $I(t)$.

Solution - The ODE that describes this system is:

$$5I' + 25I = 0.$$

We can rewrite this as:

$$I' + 5I = 0.$$

This is a first-order linear ODE, and the corresponding integrating factor is $\rho(t) = e^{\int 5dt} = e^{5t}$. Multiplying both sides of the ODE by this integrating factor we get:

$$e^{5t}I' + 5e^{5t}I = 0 \Rightarrow \frac{d}{dt}\left(e^{5t}I\right) = 0.$$

Integrating both sides of the above equation we get:

$$e^{5t}I = C \Rightarrow I(t) = Ce^{-5t}.$$

If we plug in the initial condition $I(0) = 4 = C$ we get the final solution:

$$I(t) = 4e^{-5t}.$$

3.7.5 - In the circuit from Problem 3.7.1, with the switch in position 1, suppose that $E(t) = 100e^{-10t}\cos 60t$, $R = 20$, $L = 2$, and $I(0) = 0$. Find $I(t)$.

Solution - The differential equation governing this system will be:

$$2I' + 20I = 100e^{-10t}\cos(60t).$$

If we divide both sides by 2 we get the ODE:

$$I' + 10I = 50e^{-10t}\cos(60t).$$

The integrating factor for this first-order linear ODE will be $\rho(t) = e^{\int 10dt} = e^{10t}$. Multiplying both sides of our equation by this integrating factor we get:

$$\frac{d}{dt}\left(e^{10t}I\right) = 50\cos(60t).$$

Integrating both sides of this equation gives us:

$$e^{10t}I = \frac{5}{6}\sin(60t) + C$$

$$\Rightarrow I(t) = \frac{5}{6}e^{-10t}\sin(60t) + Ce^{-10t}.$$

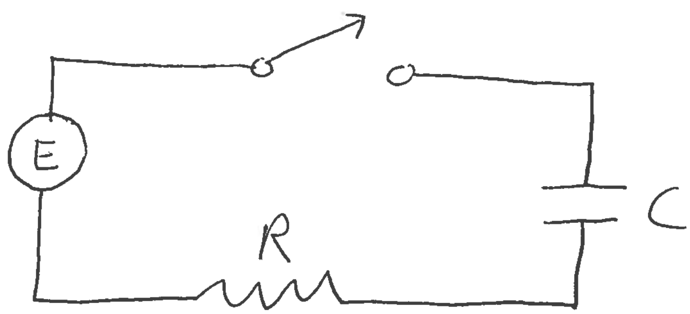
If we plug in the initial condition $I(0) = 0$ we get $C = 0$, and our solution is:

$$I(t) = \frac{5}{6}e^{-10t}\sin(60t).$$

3.7.10 - This problem deals with an RC circuit pictured below, containing a resistor (R ohms), a capacitor (C farads), a switch, a source of emf, but no inductor. This system is governed by the linear first-order differential equation

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t).$$

for the charge $Q = Q(t)$ on the capacitor at time t. Note that $I(t) = Q'(t)$.



Suppose an emf of voltage $E(t) = E_0 \cos \omega t$ is applied to the RC circuit at time $t = 0$ (with the switch closed), and $Q(0) = 0$. Substitute $Q_{sp}(t) = A\cos\omega t + B\sin\omega t$ in the differential equation to show that the steady periodic charge on the capacitor is

$$Q_{sp}(t) = \frac{E_0 C}{\sqrt{1+\omega^2 R^2 C^2}} \cos(\omega t - \beta)$$

where $\beta = \tan^{-1}(\omega RC)$.

Solution - Our steady periodic charge will be of the form:

$$Q_{sp} = A\cos(\omega t) + B\sin(\omega t).$$

Its derivative will is:

$$Q'_{sp} = -\omega A\sin(\omega t) + \omega B\cos(\omega t).$$

If we plug these values into the ODE:

$$RQ'_{sp} + \frac{1}{C}Q_{sp} = E_0\cos(\omega t),$$

we get:

$$-AR\omega\sin(\omega t) + BR\omega\cos(\omega t) + \frac{A}{C}\cos(\omega t) + \frac{B}{C}\sin(\omega t) = E_0\cos(\omega t).$$

Grouping like terms together and multiplying both sides by C we have:

$$(B - A\omega RC)\sin(\omega t) + (A + B\omega RC)\cos(\omega t) = CE_0\cos(\omega t).$$

Equating coefficients we get $B - A\omega RC = 0$, which means $B = A\omega RC$. Plugging these into the above equation gives us:

$$A(1 + \omega^2 R^2 C^2)\cos(\omega t) = CE_0\cos(\omega t).$$

So, $A = \dfrac{E_0 C}{1 + \omega^2 R^2 C^2}$, and $B = \dfrac{E_0 C^2 \omega R}{1 + \omega^2 R^2 C^2}$. From this we get:

$$Q_{sp} = \frac{E_0 C}{1+\omega^2 R^2 C^2}\left(\cos\left(\omega t\right) + \omega RC \sin\left(\omega t\right)\right).$$

If we define $\alpha = \tan^{-1}(\omega RC)$ then

$$\sin\alpha = \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}},$$

$$\cos\alpha = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}}.$$

Using these we can rewrite Q_{sp} as:

$$Q_{sp} = \frac{E_0 C}{\sqrt{1+\omega^2 R^2 C^2}}\left(\cos\left(\alpha\right)\cos\left(\omega t\right) + \sin\left(\alpha\right)\sin\left(\omega t\right)\right).$$

If we use the trigonometric identity

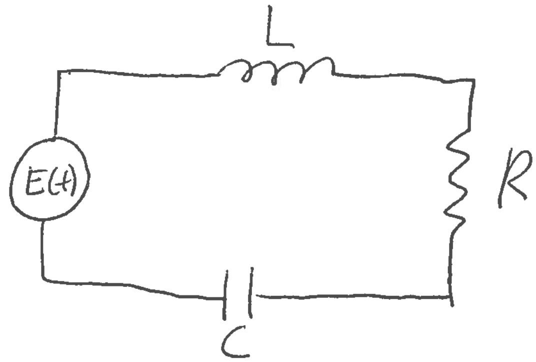
$$\cos\left(\theta - \phi\right) = \cos\left(\theta\right)\cos\left(\phi\right) + \sin\left(\theta\right)\sin\left(\phi\right)$$

we can rewrite the above equation as:

$$Q_{sp} = \frac{E_0 C}{\sqrt{1+\omega^2 R^2 C^2}}\cos\left(\omega t - \alpha\right).$$

This is our desired form.

3.7.17 For the RLC circuit pictured below find the current $I(t)$ using the given values of R, L, C and $V(t)$, and the given initial values.



$$R = 16\Omega, L = 2H, C = .02F;$$

$$E(t) = 100V; I(0) = 0, Q(0) = 5.$$

Solution - The differential equation that governs this system is:

$$2I' + 16I + 50Q = 100.$$

We can rewrite this as:

$$I' + 8I + 25Q = 50.$$

The corresponding homogeneous equation is:

$$I' + 8I + 25Q = 0.$$

The characteristic polynomial for this linear homogeneous ODE is:

$$r^2 + 8r + 25,$$

which has roots:

$$\frac{-8 \pm \sqrt{8^2 - 4(1)(25)}}{2} = -4 \pm 3i.$$

So, the corresponding homogeneous solution is:

$$Q_h(t) = c_1 e^{-4t} \cos(3t) + c_2 e^{-4t} \sin(3t).$$

As for a particular solution, we guess our particular solution is of the form $Q_p(t) = A$. Plugging this into our ODE we get:

$$25A = 50,$$

so $A = 2$. Therefore the solution to our differential equation is:

$$Q(t) = c_1 e^{-4t} \cos(3t) + c_2 e^{-4t} \sin(3t) + 2.$$

Its derivative is:

$$I(t) =$$
$$-3c_1 e^{-4t} \sin(3t) - 4c_1 e^{-4t} \cos(3t) + 3c_2 e^{-4t} \cos(3t) - 4c_2 e^{-4t} \sin(3t).$$

Plugging in our initial conditions gives us:

$$Q(0) = c_1 + 2 = 5,$$

$$I(0) = -4c_1 + 3c_2 = 0.$$

Solving this system of equations gives us $c_1 = 3, c_2 = 4$. So, our solution is:

$$Q(t) = 3e^{-4t}\cos(3t) + 4e^{-4t}\sin(3t) + 2,$$

and

$$I(t) = -25e^{-4t}\sin(3t).$$

3.7.19 Same instructions as Problem 3.7.17, but with the values:

$$R = 60\Omega, L = 2H, C = .0025F;$$

$$E(t) = 100e^{-10t}V; I(0) = 0, Q(0) = 1.$$

Solution - The differential equation that governs this circuit is:

$$2I' + 60I + 400Q = 100e^{-10t}.$$

We can rewrite this ODE as:

$$I' + 30I + 200Q = 50e^{-10t}.$$

The corresponding homogeneous equation is:

$$I' + 30I + 200Q = 0.$$

This has characteristic polynomial:

$$r^2 + 30r + 200 = (r + 20)(r + 10).$$

The roots of this polynomial are: $r = -10, -20$, and so the homogeneous solution is:

$$Q_h = c_1e^{-20t} + c_2e^{-10t}.$$

As for a particular solution, our first guess is the particular solution will be of the form Ae^{-10t}, but this would not be linearly independent of our homogeneous solution, so we need to use the guess Ate^{-10t}. With this guess we get:

$$Q_p = Ate^{-10t},$$

$$Q_p' = A(-10te^{-10t} + e^{-10t}),$$

$$Q_p'' = A(100te^{-10t} - 20e^{-10t}).$$

Plugging these into the ODE we get:

$$100Ate^{-10t} - 20Ae^{-10t} - 300Ate^{-10t} + 30Ae^{-10t} + 200Ate^{-10t} = \\ 10Ae^{-10t} = 50e^{-10t}$$

$$\Rightarrow A = 5.$$

So, our solution is:

$$Q(t) = 5te^{-10t} + c_1e^{-20t} + c_2e^{-10t}.$$

Its derivative is:

$$Q'(t) = I(t) = -50te^{-10t} + 5e^{-10t} - 20c_1e^{-20t} - 10c_2e^{-10t}.$$

Plugging in our initial conditions we get:

$$Q(0) = c_1 + c_2 = 1,$$

$$I(0) = 5 - 20c_1 - 10c_2 = 0.$$

Solving this system we get $c_1 = \frac{3}{2}, c_2 = -\frac{1}{2}$, and our current will be:

$$I(t) = 10e^{-20t} - 10e^{-10t} - 50te^{-10t}.$$