Math 2280 - Assignment 4

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Section 3.2 - 1, 10, 16, 24, 31

Section 3.3 - 1, 10, 25, 30, 43

Section 3.2 - General Solutions of Linear Equations

3.2.1 Show directly that the given functions are linearly dependent on the real line. That is, find a non-trivial linear combination of the given functions that vanishes identically.

$$f(x) = 2x, \quad g(x) = 3x^2, \quad h(x) = 5x - 8x^2.$$

Solution - The linear combination

$$6h(x) - 15f(x) + 16g(x) = 30x - 48x^2 - 30x + 48x^2 = 0$$

does the job. So, the functions are linearly dependent.

3.2.10 Use the Wronskian to prove that the given functions are linearly independent.

$$f(x) = e^x, \quad g(x) = x^{-2}, \quad h(x) = x^{-2}\ln x; x > 0.$$

Solution - The Wronskian of these three functions is:

$$W(f,g,h) = \begin{vmatrix} e^x & \frac{1}{x^2} & \frac{\ln x}{x^2} \\ e^x & -\frac{2}{x^3} & \frac{1}{x^3} - \frac{2\ln x}{x^3} \\ e^x & \frac{6}{x^4} & -\frac{5}{x^4} + \frac{6\ln x}{x^4} \end{vmatrix} = e^x\left(\frac{1}{x^5} + \frac{5}{x^6} + \frac{4}{x^7}\right).$$

For $x > 0$ this is always positive. Note we could also just plug in $x = 1$ and check for if the Wronskian is 0 or not:

$$\begin{vmatrix} e & 1 & 0 \\ e & -2 & 1 \\ e & 6 & -5 \end{vmatrix} = 10e \neq 0.$$

3.2.16 Find a particular solution to the third-order homogeneous linear equation given below, using the three linearly independent solutions given below.

$$y^{(3)} - 5y'' + 8y' - 4y = 0;$$

$$y(0) = 1, \quad y'(0) = 4, \quad y''(0) = 0;$$

$$y_1 = e^x, \quad y_2 = e^{2x}, \quad y_3 = xe^{2x}.$$

Solution - The general solution is:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}.$$

Its derivatives are:

$$y'(x) = c_1 e^x + 2c_2 e^{2x} + 2c_3 x e^{2x} + c_3 e^{2x},$$

$$y''(x) = c_1 e^x + 4c_2 e^{2x} + 4c_3 x e^{2x} + 4c_3 e^{2x}.$$

Plugging in our initial conditions we have:

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = c_1 + 2c_2 + c_3 = 4$$

$$y''(0) = c_1 + 4c_2 + 4c_3 = 0.$$

Solving this system of linear equations gives us $c_1 = -12, c_2 = 13, c_3 = -10$, and so the solution to our initial value problem is:

$$y(x) = -12e^x + 13e^{2x} - 10xe^{2x}.$$

3.2.24 Find a solution satisfying the given initial conditions for the differential equation below. A complementary solution y_c, and a particular solution y_p are given.

$$y'' - 2y' + 2y = 2x;$$

$$y(0) = 4 \quad y'(0) = 8;$$

$$y_c = c_1 e^x \cos x + c_2 e^x \sin x \quad y_p = x + 1.$$

Solution - The general solution will be:

$$y(x) = c_1 e^x \cos x + c_2 e^x \sin x + x + 1.$$

Its derivative is:

$$y'(x) = -c_1 e^x \sin x + c_1 e^x \cos x + c_2 e^x \cos x + c_2 e^x \sin x + 1.$$

If we plug in our initial conditions we get:

$$y(0) = c_1 + 1 = 4,$$

$$y'(0) = c_1 + c_2 + 1 = 8.$$

From these we get $c_1 = 3$ and $c_4 = 4$. The solution to our initial value problem is then:

$$y(x) = 3e^x \cos x + 4e^x \sin x + x + 1.$$

3.2.31 This problem indicates why we can impose *only* n initial conditions on a solution of an nth-order linear differential equation.

(a) Given the equation

$$y'' + py' + qy = 0,$$

explain why the value of $y''(a)$ is determined by the values of $y(a)$ and $y'(a)$.

(b) Prove that the equation

$$y'' - 2y' - 5y = 0$$

has a solution satisfying the conditions

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = C,$$

if and only if $C = 5$.

Solution -

(a) - $y''(a) = -p(a)y'(a) - q(a)y(a)$. So, if we know $p(a), y'(a), q(a)$, and $y(a)$, then $y''(a)$ is set.

(b) - $y''(0) = 2y'(0) + 5y'(0) = 2(0) + 5(1) = 5$. So, we must have $C = 5$.

Section 3.3 - Homogeneous Equations with Constant Coefficients

3.3.1 - Find the general solution to the differential equation

$$y'' - 4y = 0.$$

Solution - The characteristic polynomial is:

$$r^2 - 4 = (r+2)(r-2).$$

This polynomial has roots $r = \pm 2$, and so the general solution is:

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

3.3.10 - Find the general solution to the differential equation

$$5y^{(4)} + 3y^{(3)} = 0.$$

Solution - The characteristic polynomial is:

$$5r^4 + 3r^3 = 5r^3\left(r + \frac{3}{5}\right).$$

The roots of this polynomial are $r = 0, 0, 0, -\frac{3}{5}$. So, the general solution is:

$$y(x) = c_1 + c_2x + c_3x^2 + c_4e^{-\frac{3}{5}x}.$$

3.3.25 - Solve the initial value problem

$$3y^{(3)} + 2y'' = 0;$$
$$y(0) = -1, \quad y'(0) = 0, \quad y''(0) = 1.$$

Solution - The characteristic polynomial for this ODE is:

$$3r^3 + 2r^2 = 3r^2\left(r + \frac{2}{3}\right).$$

This polynomial has roots $r = 0, 0, -\frac{2}{3}$. So, the general solution is:

$$y(x) = c_1 + c_2 x + c_3 e^{-\frac{2}{3}x}.$$

Its derivatives are:

$$y'(x) = c_2 - \frac{2}{3}c_3 e^{-\frac{2}{3}x},$$
$$y''(x) = \frac{4}{9}c_3 e^{-\frac{2}{3}x}.$$

If we plug in the initial conditions we get:

$$y''(0) = \frac{4}{9}c_3 = 1 \Rightarrow c_3 = \frac{9}{4},$$
$$y'(0) = c_2 - \frac{3}{2} = 0 \Rightarrow c_2 = \frac{3}{2},$$
$$y(0) = c_1 + \frac{9}{4} = -1 \Rightarrow c_1 = -\frac{13}{4}.$$

So, the unique solution to the initial value problem is:

$$y(x) = -\frac{13}{4} + \frac{3}{2}x + \frac{9}{4}e^{-\frac{2}{3}x}.$$

3.3.30 - Find the general solution to the differential equation

$$y^{(4)} - y^{(3)} + y'' - 3y' - 6y = 0.$$

Solution - The characteristic polynomial is:

$$r^4 - r^3 + r^2 - 3r - 6 = (r+1)(r-2)(r^2+3).$$

The roots of this polynomial are $r = -1, 2, \pm 3i$. Therefore the general solution is:

$$y(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos{(3x)} + c_4 \sin{(3x)}.$$

3.3.43 -

(a) - Use Euler's formula to show that every complex number can be written in the form $re^{i\theta}$, where $r \geq 0$ and $-\pi < \theta \leq \pi$.

(b) - Express the numbers 4, -2, $3i$, $1+i$, and $-1+i\sqrt{3}$ in the form $re^{i\theta}$.

(c) - The two square roots of $re^{i\theta}$ are $\pm\sqrt{r}e^{i\theta/2}$. Find the square roots of the numbers $2-2i\sqrt{3}$ and $-2+2i\sqrt{3}$.

Solution -

(a) - We can write any complex number $a+ib$ as $r\cos\theta + ir\sin\theta$, where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, where $\tan^{-1}$ is treated as a function with two inputs, the numerator and the denominator, instead of just the single input of their quotient.

(b) -

$$4 = 4,$$
$$-2 = 2e^{i\pi},$$
$$3i = 3e^{i\frac{\pi}{2}},$$
$$1+i = \sqrt{2}e^{i\frac{\pi}{4}},$$
$$-1+i\sqrt{3} = 2e^{i\frac{2\pi}{3}}.$$

(c) -

$$\sqrt{2-2i\sqrt{3}} = \pm 8^{\frac{1}{4}}e^{-i\frac{\pi}{6}},$$
$$\sqrt{-2+2i\sqrt{3}} = \pm 8^{\frac{1}{4}}e^{i\frac{\pi}{3}}.$$