Math 2280 - Assignment 2

Dylan Zwick

Spring 2014

Section 1.5 - 1, 15, 21, 29, 38, 42

Section 1.6 - 1, 3, 13, 16, 22, 26, 31, 36, 56

Section 2.1 - 1, 8, 11, 16, 29

Section 2.2 - 1, 10, 21, 23, 24

Section 1.5 - Linear First-Order Equations

1.5.1 Find the solution to the initial value problem

$$y' + y = 2 \qquad y(0) = 0$$

Solution - The integrating factor will be:

$$\rho(x) = e^{\int 1 dx} = e^x.$$

Multiplying both sides by this integrating factor we get:

$$e^x y' + e^x y = 2e^x$$

$$\Rightarrow \frac{d}{dx}(e^x y) = 2e^x.$$

Taking the antiderivative of both sides of the equation we get:

$$e^x y = 2e^x + C.$$

Solving this for y:

$$y(x) = 2 + Ce^{-x}.$$

Plugging in the initial condition $y(0) = 0$ and solving for C:

$$0 = 2 + Ce^{-0} \rightarrow 0 = 2 + C \rightarrow C = -2.$$

So, our answer is:

$$y(x) = 2 - 2e^{-x}.$$

1.5.15 Find the solution to the initial value problem

$$y' + 2xy = x, \qquad y(0) = -2.$$

Solution - The integrating factor will be:

$$\rho(x) = e^{\int 2x\,dx} = e^{x^2}.$$

Multiplying both sides of the differential equation by this integrating factor gives us:

$$e^{x^2} y' + 2xe^{x^2} y = xe^{x^2}$$

$$\Rightarrow \frac{d}{dx}(e^{x^2} y) = xe^{x^2}.$$

Taking the antiderivative of both sides gives us:

$$e^{x^2} y = \frac{1}{2}e^{x^2} + C.$$

Solving for $y(x)$:

$$y(x) = Ce^{-x^2} + \frac{1}{2}.$$

Plugging in the initial condition $y(0) = -2$ and solving for C we get:

$$-2 = Ce^{-0^2} + \frac{1}{2} \rightarrow C = -\frac{5}{2}.$$

So, the answer is:

$$y(x) = \frac{1 - 5e^{-x^2}}{2}.$$

3

1.5.21 Find the solution to the initial value problem

$$xy' = 3y + x^4 \cos x, \qquad y(2\pi) = 0.$$

Solution - Dividing both sides of the differential equation by x, and doing a bit of algebra, we get:

$$y' - \frac{3}{x}y = x^3 \cos x.$$

The integrating factor will be:

$$\rho(x) = e^{-\int \frac{3}{x} dx} = e^{-3 \ln x} = x^{-3} = \frac{1}{x^3}.$$

Multiplying both sides of the differential equation by this integrating factor we get:

$$\frac{1}{x^3}y' - \frac{3}{x^4}y = \cos x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x^3}\right) = \cos x.$$

Taking the antiderivative of both sides gives us:

$$\frac{y}{x^3} = \sin x + C.$$

Solving this for $y(x)$:

$$y(x) = x^3 \sin x + Cx^3.$$

Plugging in the initial condition $y(2\pi) = 0$ and solving for C we get:

$$0 = (2\pi)^3 \sin(2\pi) + C(2\pi)^2 \rightarrow 0 = 4\pi^2 C \rightarrow C = 0.$$

So, our answer is:

$$y(x) = x^3 \sin x.$$

1.5.29 Express the general solution of $dy/dx = 1 + 2xy$ in terms of the *error function*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Solution - We can rewrite the differential equation

$$\frac{dy}{dx} = 1 + 2xy$$

as

$$y' - 2xy = 1.$$

The integrating factor for this first-order linear ODE will be:

$$\rho(x) = e^{-\int 2x dx} = e^{-x^2}.$$

Multiplying both sides of the differential equation by this integrating factor gives us:

$$e^{-x^2} y' - 2xe^{-x^2} y = e^{-x^2}$$

$$\Rightarrow \frac{d}{dx}(e^{-x^2} y) = e^{-x^2}.$$

Taking the antiderivative of both sides gives us:

$$e^{-x^2} y = \int e^{-x^2} dx = \frac{\sqrt{\pi}}{2} erf(x) + C.$$

Solving this for $y(x)$ we get:

$$y(x) = e^{x^2} \left(\frac{\sqrt{\pi}}{2} erf(x) + C \right).$$

6

1.5.38 Consider the *cascade* of two tanks shown below with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volumes of brine in the two tanks. Each tank also initially contains 50 lbs of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank 1.
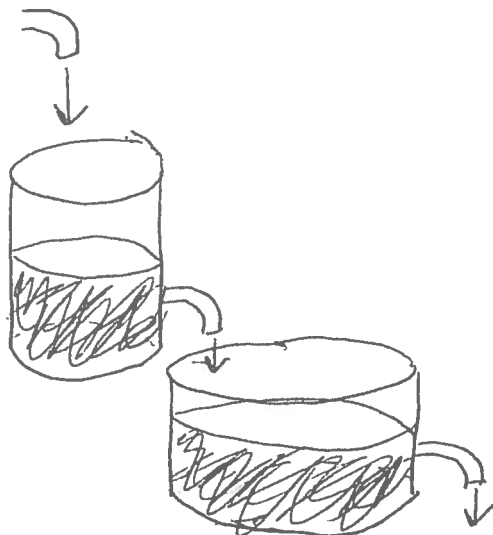
(a) Find the amount $x(t)$ of salt in tank 1 at time t.

If we assume instantaneous mixing we have,
if x is the amount of salt, then

$$\frac{dx}{dt} = -x\left(\frac{5}{100}\right)$$

$$\Rightarrow \quad x(t) = Ce^{-\frac{5}{100}t}$$

$$x(0) = 50 \Rightarrow C = 50,$$

$$x(t) = 50\, e^{-\frac{5}{100}t} = \boxed{50\, e^{-t/20}}$$

(b) Suppose that $y(t)$ is the amount of salt in tank 2 at time t. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}.$$

and then solve for $y(t)$, using the function $x(t)$ found in part (a).

Solution - Tank 2 gains $\frac{5x}{100}\Delta t$ pounds of salt in time Δt from tank 1 and, assuming instant mixing, loses $\frac{5y}{200}\Delta t$ pounds of salt in time Δt. Taking the limit as $\Delta t \to 0$ we get

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200}.$$

Using the result $x(t) = 50e^{-\frac{t}{20}}$ from part (a) we can rewrite this as

$$\frac{dy}{dt} + \frac{y}{40} = \frac{5x}{100} = \frac{5}{2}e^{-\frac{t}{20}}.$$

The integrating factor will be $e^{\int \frac{1}{40}dt} = e^{\frac{t}{40}}$. Multiplying both sides by this integrating factor we get:

$$e^{\frac{t}{40}}y' + \frac{1}{40}e^{\frac{t}{40}}y = \frac{5}{2}e^{-\frac{t}{40}}$$

$$\Rightarrow \frac{d}{dt}(ye^{\frac{t}{40}}) = \frac{5}{2}e^{-\frac{t}{40}}.$$

Taking the antiderivative of both sides:

$$ye^{\frac{t}{40}} = C - 100e^{-\frac{t}{40}}.$$

Solving this for $y(t)$ we get:

8

$$y(t) = Ce^{-\frac{t}{40}} - 100e^{-\frac{t}{20}}.$$

Using the initial condition $y(0) = 50$ and solving for C gives us:

$$50 = C - 100 \rightarrow C = 150.$$

So, our answer is:

$$y(t) = 150e^{-\frac{t}{40}} - 100e^{-\frac{t}{20}}.$$

(c) Finally, find the maximum amount of salt ever in tank 2.

Solution - We want to find where the derivative of $y(t)$ is zero.

$$y'(t) = -\frac{15}{4}e^{-\frac{t}{40}} + 5e^{-\frac{t}{20}} = 0$$

$$\Rightarrow e^{-\frac{t}{20}} = \frac{3}{4}e^{-\frac{t}{40}} \rightarrow e^{-\frac{t}{40}} = \frac{3}{4}$$

$$\rightarrow t = 40\ln\left(\frac{4}{3}\right).$$

Plugging $40\ln\left(\frac{4}{3}\right)$ in for t we get:

$$y(40\ln\left(\frac{4}{3}\right)) = 150e^{-\ln\left(\frac{4}{3}\right)} - 100e^{-2\ln\left(\frac{4}{3}\right)}) =$$

$$150\left(\frac{3}{4}\right) - 100\left(\frac{9}{16}\right) = \frac{225}{4} = 56.25 \text{ lbs. of salt.}$$

1.5.42 Suppose that a falling hailstone with density $\delta = 1$ starts from rest with negligible radius $r = 0$. Thereafter its radius is $r = kt$ (k is a constant) as it grows by accreation during its fall. Use Newton's second law - according to which the net force F acting on a possibly variable mass m equals the time rate of change dp/dt of its momentum $p = mv$ - to set up and solve the initial value problem

$$\frac{d}{dt}(mv) = mg, \qquad v(0) = 0,$$

where m is the variable mass of the hailstone, $v = dy/dt$ is its velocity, and the positive y-axis points downward. Then show that $dv/dt = g/4$. Thus the hailstone falls as though it were under *one-fourth* the influence of gravity.

Solution - The product rule tells us:

$$\frac{d}{dt}(mv) = \frac{dm}{dt}v + m\frac{dv}{dt}.$$

The mass of the hailstone as a function of time is:

$$m(t) = \delta\left(\frac{4}{3}\pi r(t)^3\right)$$

$$\Rightarrow \frac{dm}{dt} = 4\delta\pi r^2 \frac{dr}{dt}.$$

Now, $\dfrac{dr}{dt} = k$, so

$$\frac{dm}{dt} = \frac{3mk}{r}.$$

Plugging this in for $\dfrac{dm}{dt}$ in the product rule equation, and using that $\dfrac{d}{dt}(mv) = mg$, we get:

$$m\frac{dv}{dt} + \frac{3mk}{r}v = mg.$$

As $r(t) = kt$ this becomes

$$m\frac{dv}{dt} + \frac{3m}{t}v = mg.$$

Dividing everything by m we get:

$$\frac{dv}{dt} + \frac{3}{t}v = g.$$

Multiplying both sides by the integrating factor

$$\rho(t) = e^{\int \frac{3}{t}dt} = e^{3\ln t} = t^3$$

we get:

$$t^3\frac{dv}{dt} + 3t^2v = t^3g.$$

$$\Rightarrow \frac{d}{dt}(t^3v) = t^3g.$$

Integrating both sides and solving for v we get:

$$t^3 v = \frac{t^4}{4}g + C$$

$$\Rightarrow v(t) = \frac{t}{4}g + \frac{C}{t^3}.$$

Now, there's a singularity at $t = 0$. If we assume that $v(t) \to 0$ as $t \to 0^+$ we must have $C = 0$. So,

$$v(t) = \frac{t}{4}g.$$

Differentiating this we get

$$\frac{dv}{dt} = \frac{g}{4},$$

which is what we wanted to prove.

Note that what's going on here is that our model says the radius of the hailstone is kt, so at time $t = 0$ the radius is zero, and the hailstone doesn't exist. So, it's not too surprising there's a singularity at $t = 0$. So, we really want to restrict ourselves to positive times, that is to say, times at which the hailstone exists, and treat the initial condition $v(0) = 0$ as really being a statement that the limit of $v(t)$ at $t \to 0^+$ is 0.

Section 1.6 - Substitution Methods and Exact Equations

1.6.1 Find the general solution of the differential equation

$$(x + y)y' = x - y$$

Solution - We can rewrite the above differential equation as:

$$(x + y)\frac{dy}{dx} = x - y.$$

Multiplying both sides by dx gives us:

$$(x + y)dy = (x - y)dx$$
$$\Rightarrow (x + y)dy + (y - x)dx = 0.$$

If we define $M(x, y) = y - x$ and $N(x, y) = x + y$ then a quick check:

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x},$$

verifies the differential equation is exact. Integrating $M(x, y)$ with respect to x gives us:

$$\int M dx = \int (y - x)dx = xy - \frac{x^2}{2} + g(y) = F(x, y).$$

Taking the partial derivative of $F(x, y)$ with respect to y we get:

$$\frac{\partial F}{\partial y} = x + g'(y) = x + y.$$

So, $g'(y) = y$, which means $g(y) = \dfrac{y^2}{2}$, and our solution is:

$$F(x, y) = xy - \frac{x^2}{2} + \frac{y^2}{2} = C.$$

1.6.3 Find the general solution of the differential equation

$$xy' = y + 2\sqrt{xy}$$

Solution - If we divide both sides of the above differential equation by x we get:

$$y' = \frac{y}{x} + 2\sqrt{\frac{y}{x}}.$$

This is a homogeneous equation. If we substitute $v = \frac{y}{x}$ we get $y = xv$, and consequently $y' = xv' + v$, so the differential equation becomes:

$$xv' + v = v + 2\sqrt{v}.$$

We can rewrite the differential equation directly above as:

$$x\left(\frac{dv}{dx}\right) = 2\sqrt{v}.$$

This is a separable differential equation, which we can write as:

$$\frac{dv}{\sqrt{v}} = \frac{2dx}{x}.$$

Integrating both sides of the above equation we get:

$$2\sqrt{v} = 2\ln x + C$$

$$\Rightarrow \sqrt{v} = \ln x + C$$

$$\Rightarrow v = (\ln x + C)^2.$$

If we plug back in $v = \dfrac{y}{x}$ we get:

$$\frac{y}{x} = (\ln x + C)^2$$

$$\Rightarrow y(x) = x(\ln x + C)^2.$$

1.6.13 Find the general solution of the differential equation

$$xy' = y + \sqrt{x^2 + y^2}$$

Hint - You may find the following integral useful:

$$\int \frac{dv}{\sqrt{1 + v^2}} = \ln\left(v + \sqrt{1 + v^2}\right).$$

Solution - If we divide everything in the above differential equation by x we get:

$$y' = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}.$$

This is a homogeneous equation, and so we make the substitutuion $v = \frac{y}{x}$, which implies $y' = xv' + v$. Plugging these into the equation above we get:

$$xv' + v = v + \sqrt{1 + v^2},$$

$$\Rightarrow xv' = \sqrt{1 + v^2},$$

$$\Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}.$$

Integrating both sides of the above equation[1] we get:

$$\ln\left(v + \sqrt{1 + v^2}\right) = \ln x + C,$$

$$\Rightarrow v + \sqrt{1 + v^2} = Cx.$$

---

[1] And using the suggested integral.

Plugging in $v = \dfrac{y}{x}$ and multiplying both sides by x we get our solution:

$$y + \sqrt{x^2 + y^2} = Cx^2.$$

1.6.16 Find the general solution of the differential equation

$$y' = \sqrt{x + y + 1}$$

Solution - If we make the substitution $v = x + y + 1$ we get $v' = 1 + y'$, and our equation becomes:

$$v' - 1 = \sqrt{v}.$$

This is a separable differential equation which we can rewrite as:

$$\frac{dv}{\sqrt{v} + 1} = dx.$$

The integral:

$$\int \frac{dv}{\sqrt{v} + 1}$$

can be solved first with the u-substitution $u = \sqrt{v}$, and so $du = \frac{dv}{2\sqrt{v}}$, which gives us the integral:

$$\int \frac{2u\,du}{u + 1} = 2\int \frac{u\,du}{u + 1}.$$

If we make the substitution $w = u + 1, dw = du$ this integral becomes:

$$2\int \frac{(w - 1)dw}{w} = 2\int dw - 2\int \frac{dw}{w} = 2w - 2\ln w.$$

20

Substituting back $w = u + 1 = \sqrt{v} + 1$ this becomes:

$$2(\sqrt{v} + 1) - 2\ln\left(\sqrt{v} + 1\right).$$

The integral on the other side is trivial:

$$\int dx = x + C.$$

So, our solution is:

$$2(\sqrt{v} + 1) - 2\ln\left(\sqrt{v} + 1\right) = x + C.$$

If we plug in $v = x + y$ and do a little algebra this becomes:

$$x = 2\sqrt{x + y + 1} - 2\ln\left(1 + \sqrt{x + y + 1}\right) + C.$$

1.6.22 Find the general solution of the differential equation

$$x^2 y' + 2xy = 5y^4$$

Solution - We can rewrite the above differential equation as:

$$y' + \frac{2}{x}y = \frac{5}{x^2}y^4.$$

This is a Bernoulli equation with $n = 4$. We make the substitution:

$$v = y^{1-4} = y^{-3}$$

to transform the ODE into:

$$\frac{dv}{dx} + (1-4)\left(\frac{2}{x}\right)v = (1-4)\frac{5}{x^2}$$

$$\Rightarrow \frac{dv}{dx} - \frac{6}{x}v = -\frac{15}{x^2}.$$

This is a first-order linear ODE, so we use the integrating factor

$$\rho(x) = e^{\int -\frac{6}{x}dx} = x^{-6}$$

to get

$$\frac{d}{dx}(x^{-6}v) = -\frac{15}{x^8}.$$

Integrating both sides we get:

$$x^{-6}v = \frac{15}{7}x^{-7} + C$$

$$\Rightarrow v(x) = \frac{15}{7x} + Cx^6.$$

Plugging back in $v = 1/y^3$ we get:

$$\frac{1}{y^3} = \frac{15 + Cx^7}{7x}$$

$$\Rightarrow y^3 = \frac{7x}{15 + Cx^7}.$$

1.6.26 Find the general solution of the differential equation

$$3y^2 y' + y^3 = e^{-x}$$

Solution - We can rewrite the above ODE as:

$$y' + \frac{1}{3}y = \frac{y^{-2}}{3}e^{-x}$$

which is a Bernoulli equation with $n = -2$. Making the substitution $v = y^{1-(-2)} = y^3$ we get:

$$\frac{dv}{dx} + v = e^{-x}.$$

This is a first-order linear ODE with integrating factor:

$$\rho(x) = e^{\int dx} = e^x.$$

If we multiply both sides by this integrating factor we get:

$$\frac{d}{dx}(e^x v) = 1.$$

Integrating both sides we get:

$$e^x v = x + C$$

$$\Rightarrow v = Ce^{-x} + xe^{-x}.$$

So, using $v = y^3$, we get our solution:

$$y^3 = e^{-x}(x + C).$$

1.6.31 Verify that the differential equation

$$(2x + 3y)dx + (3x + 2y)dy = 0$$

is exact; then solve it.

Solution - Setting $M = 2x + 3y$ and $N = 3x + 2y$ we have:

$$\frac{\partial M}{\partial y} = 3 = \frac{\partial N}{\partial x}.$$

So, the ODE is exact. Solving for $F(x, y)$ we get:

$$F(x, y) = \int M dx = x^2 + 3xy + g(y),$$

and

$$\frac{\partial F}{\partial y} = 3x + g'(y) = 3x + 2y.$$

So, $g'(y) = 2y$, and therefore $g(y) = y^2 + C$. So, our final solution is $F(x, y) = 0$, or:

$$x^2 + 3xy + y^2 = C.$$

1.6.36 Verify that the differential equation

$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$$

is exact; then solve it.

Solution - To verify the ODE is exact we set $M = 1 + ye^{xy}$ and $N = 2y + xe^{xy}$. Then

$$\frac{\partial M}{\partial y} = xye^{xy} + e^{xy} = \frac{\partial N}{\partial x}.$$

Solving for $F(x, y)$ we get:

$$F(x, y) = \int M dx = x + e^{xy} + g(y).$$

This means

$$\frac{\partial F}{\partial y} = xe^{xy} + g'(y) = xe^{xy} + 2y.$$

From this we get $g(y) = y^2 + C$ and therefore our solution is

$$x + e^{xy} + y^2 = C.$$

1.6.56 Suppose that $n \neq 0$ and $n \neq 1$. Show that the substitutuion $v = y^{1-n}$ transforms the Bernoulli equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

into the linear equation

$$\frac{dv}{dx} + (1-n)P(x)v(x) = (1-n)Q(x).$$

Solution - If we make the substitution $v = y^{1-n}$ then $v^{\frac{1}{1-n}} = y$ and

$$\frac{dy}{dx} = \frac{1}{1-n}v^{\left(\frac{1}{1-n}-1\right)}\left(\frac{dv}{dx}\right) = \frac{1}{1-n}v^{\frac{n}{1-n}}\frac{dv}{dx}.$$

If we note $y^n = v^{\frac{n}{1-n}}$ then

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

becomes

$$\frac{1}{1-n}v^{\frac{n}{1-n}}\frac{dv}{dx} + P(x)v^{\frac{1}{1-n}} = Q(x)v^{\frac{n}{1-n}}.$$

If we multiply both sides by $(1-n)v^{-\frac{n}{1-n}}$ we get:

$$\frac{dv}{dx} + (1-n)P(x)v^{\frac{1-n}{1-n}} = (1-n)Q(x)$$

$$\Rightarrow \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

Section 2.1 - Population Models

2.1.1 Separate variables and use partial fractions to solve the initial value problem:

$$\frac{dx}{dt} = x - x^2 \qquad\qquad x(0) = 2.$$

Solution - The differential equation above is separable:

$$\frac{dx}{x - x^2} = dt.$$

Noting $x - x^2 = x(1 - x)$ the quotient on the left we can break up as:

$$\left(\frac{A}{x} + \frac{B}{1 - x} \right) dx = \frac{(A + (B - A)x)dx}{x - x^2}.$$

So, $A = 1$, and $B - A = 0$, which implies $B = 1$, and the integral on the left becomes:

$$\int \frac{dx}{x} + \int \frac{dx}{1 - x} = \ln x - \ln (1 - x) = \ln \left(\frac{x}{1 - x} \right).$$

The integral on the left is $t + C$, and so we have:

$$\ln \left(\frac{x}{1 - x} \right) = t + C,$$

and so

$$\frac{x}{1-x} = Ce^t.$$

If we solve for $x(t)$ we get:

$$x(t) = \frac{Ce^t}{1 + Ce^t}.$$

Plugging in $x(0) = 2$ we get:

$$x(0) = \frac{C}{1+C} = 2 \Rightarrow C = -2.$$

So, our solution is:

$$x(t) = \frac{2}{2 - e^{-t}}.$$

2.1.8 Separate variables and use partial fractions to solve the initial value problem:

$$\frac{dx}{dt} = 7x(x - 13) \qquad\qquad x(0) = 17.$$

Solution - The above differential equation separates as:

$$\frac{dx}{7x(x - 13)} = dt.$$

We do a partial fraction decomposition of the quotient on the left:

$$\frac{A}{7x} + \frac{B}{x - 13} = \frac{1}{7x(x - 13)}.$$

Solving for A and B we get:

$$\frac{A(x - 13) + 7Bx}{7x(x - 13)} = \frac{(A + 7B)x - 13A}{7x(x - 13)} = \frac{1}{7x(x - 13)}.$$

From this we get $A = -\frac{1}{13}$ and $B = \frac{1}{91}$. So, our integral is:

$$\int \left(-\frac{1}{91x} + \frac{1}{91(x - 13)} \right) dx = -\frac{\ln x}{91} + \frac{\ln (x - 13)}{91} = t + C.$$

After a little algebra we get:

30

$$\ln\left(\frac{x-13}{x}\right) = 91t + C$$

$$\Rightarrow \frac{x-13}{x} = Ce^{91t}$$

$$\Rightarrow x(t) = \frac{13}{1 - Ce^{91t}}.$$

Using $x(0) = 17$ we get:

$$x(0) = \frac{13}{1 - C} = 17 \Rightarrow C = \frac{4}{17}.$$

So,

$$x(t) = \frac{13}{1 - \frac{4}{17}e^{91t}} = \frac{221}{17 - 4e^{91t}}.$$

2.1.11 Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$.

(a) Show that

$$P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2.$$

(b) If $P_0 = 100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

Solutions -

(a) - By definition we have

$$\beta = \frac{\beta_0}{\sqrt{P}} \qquad\qquad \delta = \frac{\delta_0}{\sqrt{P}}.$$

The differential equation modeling population growth is:

$$\frac{dP}{dt} = (\beta - \delta)P = (\beta_0 - \delta_0)\sqrt{P} = k\sqrt{P},$$

where $k = \beta_0 - \delta_0$. From this we get:

$$\frac{dP}{dt} = k\sqrt{P} \Rightarrow \frac{dP}{\sqrt{P}} = kdt$$

$$\Rightarrow 2\sqrt{P} = kt + C$$

$$\Rightarrow P(t) = \left(\frac{1}{2}kt + C\right)^2.$$

We note $P_0 = P(0) = C^2$, so $C = \sqrt{P_0}$, and our population function is:

$$P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2.$$

(b) If we have $P_0 = 100$ then our population function is:

$$P(t) = \left(\frac{1}{2}kt + \sqrt{100}\right)^2 = \left(\frac{1}{2}kt + 10\right)^2.$$

Plugging in $P(6) = 169$ we get:

$$P(6) = \left(\frac{1}{2}k(6) + 10\right)^2 = 169$$

$$\Rightarrow 3k + 10 = 13 \Rightarrow k = 1.$$

So, our population equation is:

$$P(t) = \left(\frac{1}{2}t + 10\right)^2.$$

The number of fish after 1 year (12 months) is:

$$P(12) = \left(\frac{1}{2}(12) + 10\right)^2 = 16^2 = 256.$$

2.1.16 Consider a rabbit population $P(t)$ satisfying the logistic equation $dP/dt = aP - bP^2$. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occuring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M?

Solution - We have

$$a(120) = 8 \Rightarrow a = \frac{8}{120} = \frac{1}{15}$$

$$b(120)^2 = 6 \Rightarrow b = \frac{6}{120^2} = \frac{1}{2400}.$$

So, our population differential equation is:

$$\frac{dP}{dt} = \frac{1}{15}P - \frac{1}{2400}P^2.$$

This is a separable differential equation, which we can write as:

$$\frac{2400dP}{160P - P^2} = dt.$$

Noting that $160P - P^2 = P(160 - P)$ a partial fraction decomposition of the quotient on the left is:

$$\frac{A}{P} + \frac{B}{160 - P} = \frac{160A + (B - A)P}{P(160 - P)} = \frac{2400}{160P - P^2}.$$

Solving for A and B we get $A = 15, B = 15$, and the integral becomes:

$$15 \int \left(\frac{1}{P} + \frac{1}{160 - P} \right) dP = 15 \ln P - 15 \ln (160 - P) =$$

$$15 \ln \left(\frac{P}{160 - P} \right).$$

If we equate this to $\int dt = t + C$ we get:

$$15 \ln \left(\frac{P}{160 - P} \right) = t + C$$

$$\Rightarrow \frac{P}{160 - P} = Ce^{\frac{t}{15}}.$$

Solving for $P(t)$ we get:

$$P(t) = \frac{160Ce^{\frac{t}{15}}}{1 + Ce^{\frac{t}{15}}} = \frac{160}{1 + Ce^{-\frac{t}{15}}}.$$

Using $P(0) = 120 = \frac{160}{1 + C}$ and solving for C we get $C = \frac{1}{3}$. So, our population function is:

$$P(t) = \frac{480}{3 + e^{-\frac{t}{15}}}.$$

As $t \to \infty$ we have $P(t) \to \frac{480}{3} = 160$. This is the limiting population, and 95% of the limiting population is 152.

If we solve for when $P(t) = 152$ we get:

$$152 = \frac{480}{3 + e^{\frac{-t_*}{15}}}$$

$$\Rightarrow e^{-\frac{t_*}{15}} = \frac{480}{152} - 3 = \frac{24}{152} = \frac{3}{19}.$$

Solving this for t_* we get:

$$t_* = -15 \ln \left(\frac{3}{19} \right) = 27.69 \text{ months.}$$

So, after 27.69 months (almost 2 and one-third year) the poplu-ation will be at 95% of its limiting population.

2.1.29 During the period from 1790 to 1930 the U.S. population $P(t)$ (t in years) grew from 3.9 million to 123.2 million. Throughout this period, $P(t)$ remained close to the solution of the initial value problem

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2, \qquad\qquad P(0) = 3.9.$$

(a) What 1930 population does this logistic equation predict?

(b) What limiting population does it predict?

(c) Has this logistic equation continued since 1930 to accurately model the U.S. population?

[This problem is based on the computation by Verhulst, who in 1845 used the 1790-1840 U.S. population data to predict accurately the U.S. population through the year 1930 (long after his own death, of course).]

Solution -

(a) - Using the logistic population formula from the textbook:

$$P(t) = \frac{(210.54)(3.9)}{3.9 + (206.64)e^{-.03135t}}.$$

So,

$$P(140) \approx 127.0 \text{ million people.}$$

(b) - $M = \dfrac{0.03135}{.0001489} = 210.54$ million.

So, about 210.5 million people.

(c) - No. The current U.S. population is above 300 million.

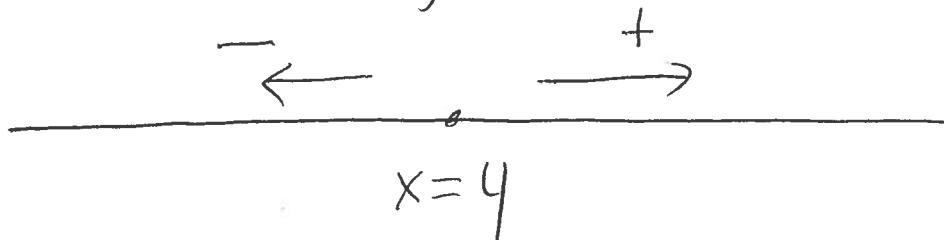# Section 2.2 - Equilibrium Solutions and Stability

2.2.1 - Find the critical points of the autonomous equation

$$\frac{dx}{dt} = x - 4.$$

Then analyze the sign of the equation to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation. Next, solve the differential equation explicitly for $x(t)$ in terms of t. Finally, use either the exact solution or a computer-generated slope field to sketch typical solution curves for the given differential equation, and verify visually the stability of each critical point.

$x - 4 = 0 \implies x = 4$ is the critical point.

Phase diagram

$x = 4$
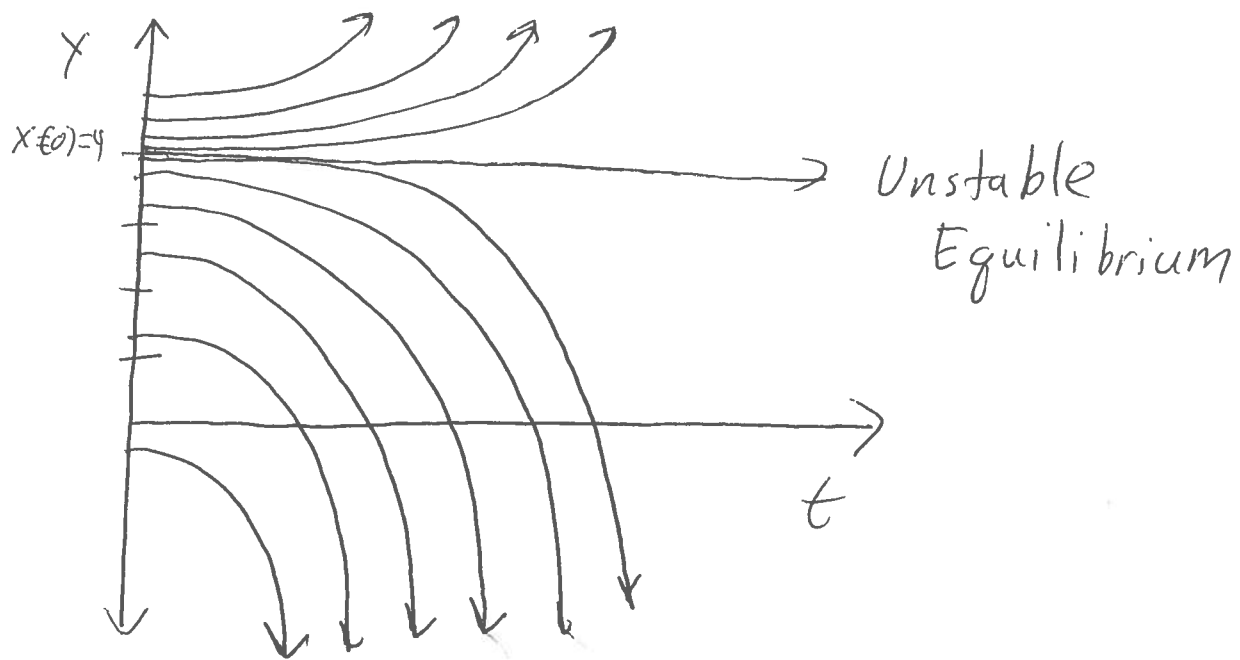
Unstable.

More space, if necessary, for problem 2.2.1.

$$\frac{dx}{x-4} = dt \implies \int \frac{dx}{x-4} = \int dt$$

$$\implies \ln(x-4) = t+C \implies x-4 = Ce^t$$

$$x(t) = Ce^t + 4$$
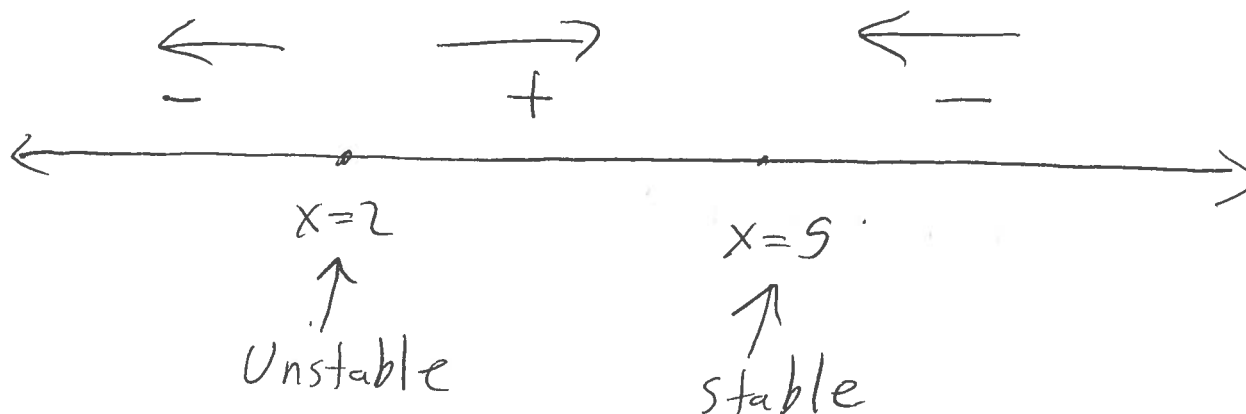

Unstable
Equilibrium

2.2.10 Find the critical points of the autonomous equation

$$\frac{dx}{dt} = 7x - x^2 - 10.$$

Then analyze the sign of the equation to determine whether each critical point is stable or unstable, and construct the corresponding phase diagram for the differential equation. Next, solve the differential equation explicitly for $x(t)$ in terms of t. Finally, use either the exact solution or a computer-generated slope field to sketch typical solution curves for the given differential equation, and verify visually the stability of each critical point.

$$7x - x^2 - 10 = (5-x)(x-2)$$

So, equilibrium points where $\frac{dx}{dt} = 0$ are
$x = 5$ and $x = 2$.

$$\xleftarrow{\quad} \qquad \xrightarrow{\quad} \qquad \xleftarrow{\quad}$$

$-$ $\qquad$ $+$ $\qquad$ $-$

$\xleftarrow{\rule{3cm}{0.4pt}\quad X=2 \quad\rule{5cm}{0.4pt}\quad X=5 \quad\rule{3cm}{0.4pt}}\to$

$X=2$
$\uparrow$
Unstable

$X=5$
$\uparrow$
stable

14

More space, if necessary, for problem 2.2.10.

$$\int \frac{dx}{(5-x)(x-2)} = \int dt$$

$$A(x-2) + B(5-x) = 1$$
$$-2A + 5B = 1$$

$$\int \left(\frac{A}{5-x} + \frac{B}{x-2} \right) dx = \int dt$$

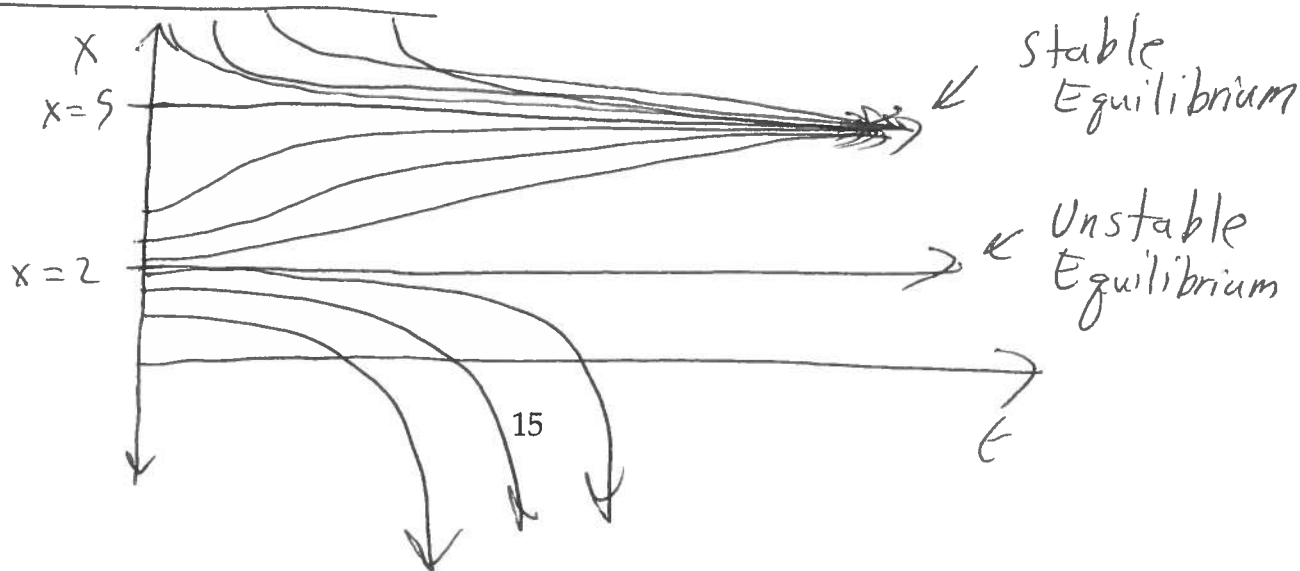$$A - B = 0 \Rightarrow A = B$$
$$A = B = \frac{1}{3}$$

$$\frac{1}{3} \int \left(\frac{1}{5-x} + \frac{1}{x-2} \right) dx = \int dt$$

$$\Rightarrow \frac{1}{3} \ln \left(\frac{x-2}{5-x} \right) = t + C \Rightarrow \frac{x-2}{5-x} = Ce^{3t}$$

$$x - 2 = (5-x) Ce^{3t} \Rightarrow \boxed{x(t) = \frac{2 + 5Ce^{3t}}{1 + Ce^{3t}}}$$
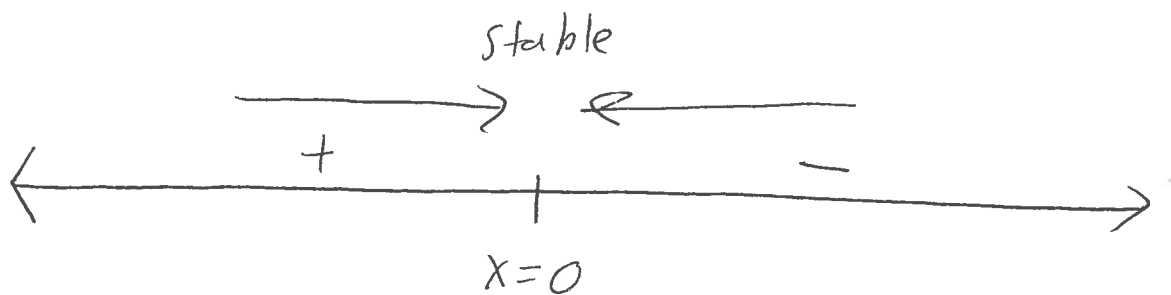
Solution Curves

x
x=5 — stable Equilibrium
x=2 — Unstable Equilibrium

15

t

2.2.21 Consider the differential equation $dx/dt = kx - x^3$.

   (a) If $k \leq 0$, show that the only critical value $c = 0$ of x is stable.

   (b) If $k > 0$, show that the critical point $c = 0$ is now unstable, but that the critical points $c = \pm\sqrt{k}$ are stable. Thus the qualitative nature of the solutions changes at $k = 0$ as the parameter k increases, and so $k = 0$ is a bifurcation point for the differential equation with parameter k.

   The plot of all points of the form (k, c) where c is a critical point of the equation $x' = kx - x^3$ is the "pitchform diagram" show in figure 2.2.13 of the textbook.

$kx - x^3 = \quad x(k-x^2)$ ~~If~~ The roots are

$x = 0$, and $x = \pm\sqrt{k}$. If $k \leq 0$ the $\pm\sqrt{k}$

1) imaginary or also 0, and we have only

one real root.

stable

$\longrightarrow$ $\leftarrow$

$\leftarrow$ $+$ $-$ $\longrightarrow$

$x = 0$

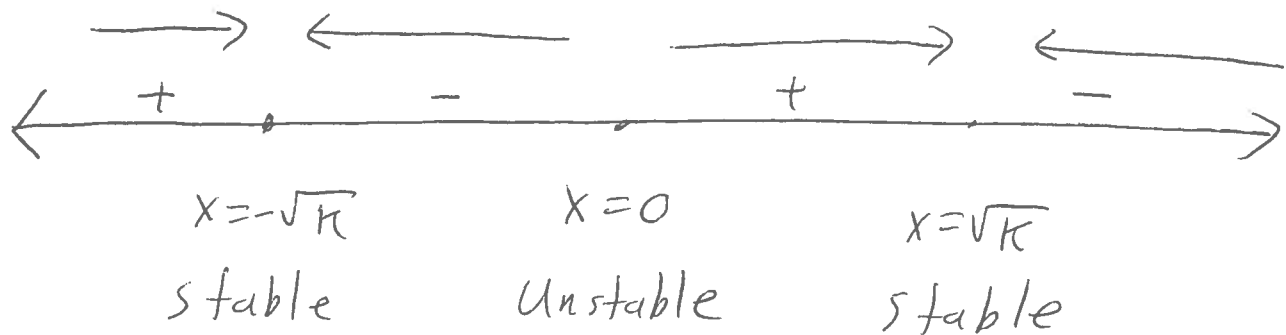$kx - x^3 < 0$ for $k \leq 0$, ~~$\neq 0$~~ $x > 0$

$kx - x^3 > 0$ for $k \leq 0$, $x < 0$

16

b) For $k > 0$ there are three distinct real roots

$$kx - x^3 = -x(x + \sqrt{k})(x - \sqrt{k})$$

~~So, as k~~ The roots are $x = 0$, $x = \pm\sqrt{k}$

$x = -\sqrt{k}$ $x = 0$ $x = \sqrt{k}$
stable Unstable stable

2.2.23 Suppose that the logistic equation $dx/dt = kx(M-x)$ models a population $x(t)$ of fish in a lake after t months during which no fishing occurs. Now suppose that, because of fishing, fish are removed from the lake at a rate of hx fish per month (with h a positive constant). Thus fish are "harvested" at a rate proportional to the existing fish population, rather than at the constant rate of Example 4 from the textbook.

(a) If $0 < h < kM$, show that the population is still logistic. What is the new limiting population?

(b) If $h \geq kM$, show that $x(t) \to 0$ as $t \to \infty$, so the lake is eventually fished out.

Solution -

(a) - If $h < kM$ then our differential equation is:

$$\frac{dx}{dt} = kx(M-x) - hx = x(kM-h) - kx^2 = kx\left(\left(M - \frac{h}{k}\right) - x\right).$$

This is still a logistic population growth equation with limiting population $M - \dfrac{h}{k}$.

(b) - If $h \geq kM$ the solution to the logistic population equation will be:

$$P(t) = \frac{\left(M - \frac{h}{k}\right) P_0}{P_0 + \left(M - \frac{h}{k} - P_0\right) e^{-k\left(M - \frac{h}{k}\right)t}}.$$

if we define $N = \dfrac{h}{k} - M \geq 0$ we can rewrite this as:

$$P(t) = \frac{-NP_0}{P_0 - (P_0 + N)e^{kNt}} = \frac{NP_0}{(P_0 + N)e^{kNt} - P_0}.$$

Taking the limit as $t \to \infty$ we get:

$$\lim_{t \to \infty} \frac{NP_0}{(P_0 + N)e^{kNt} - P_0} = 0.$$

So, the lake is eventually fished out.

Note that if $h = kM$ then our differential equation becomes

$$\frac{dx}{dt} = -kx^2.$$

The solution to this ODE is:

$$x(t) = \frac{P_0}{1 + P_0 kt}.$$

This also goes to 0 as $t \to \infty$.

2.2.24 Separate variables in the logistic harvesting equation

$$dx/dt = k(N - x)(x - H)$$

and then use partial fractions to derive the solution given in equation 15 of the textbook (also appearing in the lecture notes).

Solution - The differential equation above is separable, and can be written as:

$$\frac{dx}{(N - x)(x - H)} = k dt.$$

Integrating both sides we get:

$$\int \frac{dx}{(N - x)(x - H)} = \int k dt = kt + C.$$

For the integral on the left we take a partial fraction decomposition:

$$\frac{1}{(N - x)(x - H)} = \frac{A}{N - x} + \frac{B}{x - H} = \frac{A(x - H) + B(N - x)}{(N - x)(x - H)}.$$

So, we must have:

$$A(x - H) + B(N - x) = (A - B)x + (BN - AH) = 1.$$

From this we get $A - B = 0$, and so $A = B$. Plugging this into $BN - AH$ we have $A(N - H) = 1$, and so $A = \dfrac{1}{N - H} = B$.

Plugging these values of A and B into the partial fraction decomposition gives us:

$$\frac{1}{N-H} \int \left(\frac{1}{N-x} + \frac{1}{x-H} \right) dx = \frac{1}{N-H} \ln \left(\frac{x-H}{N-x} \right) = kt + C$$

$$\Rightarrow \ln \left(\frac{x-H}{N-x} \right) = k(N-H)t + C.$$

Exponentiating both sides we get:

$$\frac{x-H}{N-x} = Ce^{k(N-H)t}$$

$$\Rightarrow x - H = (N-x)Ce^{k(N-H)t}$$

$$\Rightarrow x(1 + Ce^{k(N-H)t}) = H + NCe^{k(N-H)t}$$

$$\Rightarrow x(t) = \frac{H + NCe^{k(N-H)t}}{1 + Ce^{k(N-H)t}}.$$

Noting

$$x(0) = x_0 = \frac{H + NC}{1 + C},$$

we can solve this for C to get:

$$C = \frac{H - x_0}{x_0 - N}.$$

Plugging this value of C into our solution above gives us:

$$x(t) = \frac{H + N \left(\frac{H-x_0}{x_0-N} \right) e^{k(N-H)t}}{1 + \left(\frac{H-x_0}{x_0-N} \right) e^{k(N-H)t}}.$$