Math 2280 - Assignment 1

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Section 1.1 - 1, 12, 15, 20, 45

Section 1.2 - 1, 6, 11, 15, 27, 35, 43

Section 1.3 - 1, 6, 9, 11, 15, 21, 29

Section 1.4 - 1, 3, 17, 19, 31, 35, 53, 68

Section 1.1 - Differential Equations and Mathematical Models

1.1.1 Verify by substitution that the given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

$$y' = 3x^2; \qquad y = x^3 + 7$$

Solution - The derivative of $y(x) = x^3 + 7$ is $3x^2$. So, the solution checks out. That was easy!

1.1.12 Verify by substitution that the given function is a solution of the given differential equation.

$$x^2y'' - xy' + 2y = 0; \qquad y_1 = x\cos(\ln x), \quad y_2 = x\sin(\ln x).$$

Solution - The first and second derivatives of y_1 are:

$$y_1(x) = x\cos(\ln x),$$

$$y_1'(x) = -\sin(\ln x) + \cos(\ln x),$$

$$y_1''(x) = -\frac{\cos(\ln x)}{x} - \frac{\sin(\ln x)}{x}.$$

Plugging these into the differential equation above we get:

$$x^2\left(-\frac{\cos(\ln x)}{x} - \frac{\sin(\ln x)}{x}\right) - x(\cos(\ln x) - \sin(\ln x)) + 2x\cos(\ln x) = 0.$$

So, that one checks out. As for $y_2(x)$ we have the first and second derivatives:

$$y_2(x) = x\sin(\ln x),$$

$$y_2'(x) = \cos(\ln x) + \sin(\ln x),$$

$$y_2''(x) = -\frac{\sin(\ln x)}{x} + \frac{\cos(\ln x)}{x}.$$

Plugging these into the differential equation above we get:

$$x^2\left(-\frac{\sin(\ln x)}{x} + \frac{\cos(lnx)}{x}\right) - x(\cos(\ln x) + \sin(\ln x)) + 2x\sin(\ln x) = 0.$$

So, that one checks out too.

1.1.15 Substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation

$$y'' + y' - 2y = 0$$

Solution - The first and second derivatives of $y = e^{rx}$ are:

$$y' = re^{rx},$$
$$y'' = r^2e^{rx}.$$

If we plug these into the differential equation we get:

$$r^2e^{rx} + re^{rx} - 2e^{rx} = (r^2 + r - 2)e^{rx} = 0.$$

As $e^{rx} \neq 0$ for any r or x, in order for the above equality to be true we must have $r^2 + r - 2 = 0$. The quadratic $r^2 + r - 2$ factors as $(r + 2)(r - 1)$, and so its roots are $r = 1$ and $r = -2$. So, those are the two values of r for which we get a solution to the differential equation.

Note - The approach used in this problem you will be seeing again many, many times throughout this course.

1.1.20 First verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition.

$$y' = x - y; \qquad y(x) = Ce^{-x} + x - 1, \qquad y(0) = 10$$

Solution - The derivative of $y(x)$ is:

$$y'(x) = -Ce^{-x} + 1.$$

We have

$$x - y = x - (Ce^{-x} + x - 1) = -Ce^{-x} + 1.$$

So, the solution checks out. Solving for C we get:

$$10 = y(0) = Ce^{-0} + 0 - 1,$$

$$\Rightarrow C = 11.$$

So, the function $y(x)$ that satisfies the differential equation and the given initial condition is:

$$y(x) = 11e^{-x} + x - 1.$$

1.1.45 Suppose a population P of rodents satisfies the differential equation $dP/dt = kP^2$. Initially, there are $P(0) = 2$ rodents, and their number is increasing at the rate of $dP/dt = 1$ rodent per month when there are $P = 10$ rodents. How long will it take for this population to grow to a hundred rodents? To a thousand? What's happening here?

Solution - This is a separable differential equation, and so we can rewrite it as:

$$\frac{dP}{P^2} = kdt.$$

Taking the indefinite integral of both sides we get:

$$-\frac{1}{P} = kt + C.$$

Solving for P, and playing a bit fast and loose with the unknown constant C, we get:

$$P(t) = \frac{1}{C - kt}.$$

Now, we need to solve for k and C. We're told the initial population of rodents is 2. So,

$$P(0) = \frac{1}{C} = 2,$$

$$\Rightarrow C = \frac{1}{2}.$$

Plugging this in for C we get:

$$P(t) = \frac{2}{1 - 2kt}.$$

We're also told when $P = 10$ that $\frac{dP}{dt} = 1$. This means:

$$1 = k(10)^2 \Rightarrow k = \frac{1}{100}.$$

Plugging $\frac{1}{100}$ in for k in our solution gives us:

$$P(t) = \frac{100}{50 - t}.$$

The population will be at 100 rodents when $t = 49$. The population will be at a thousand rodents when $t = 49.9$. What's going on here is that the solution has a vertical asymptote that goes to ∞ as $t \to 50^-$. So, this is a "Doomsday" equation.

Section 1.2 - Integrals as General and Particular Solutions

1.2.1 Find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

$$\frac{dy}{dx} = 2x + 1; \qquad y(0) = 3.$$

Solution - If we take the antiderivative of both sides of the differential equation:

$$\frac{dy}{dx} = 2x + 16$$

we get:

$$y(x) = x^2 + 16x + C.$$

Plugging in the initial condition $y(0) = 3$ we get:

$$y(0) = 0^2 + 16(0) + C = 3 \Rightarrow C = 3.$$

So, the solution is:

$$y(x) = x^2 + 16x + 3.$$

1.2.6 Find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

$$\frac{dy}{dx} = x\sqrt{x^2+9} \qquad\qquad y(-4) = 0.$$

Solution - If we take the antiderivative of both sides of the differential equation:

$$\frac{dy}{dx} = x\sqrt{x^2+9}$$

we get:

$$y(x) = \frac{1}{3}(x^2+9)^{\frac{3}{2}} + C.$$

If we plug in the initial condition $y(-4) = 0$ we get:

$$y(-4) = \frac{1}{3}((-4)^2+9)^{\frac{3}{2}} + C = \frac{1}{3}(125) + C = 0,$$

and so

$$C = -\frac{125}{3}.$$

The solution is:

$$y(x) = \frac{1}{3}(x^2+9)^{\frac{3}{2}} - \frac{1}{125}.$$

1.2.11 Find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

$$a(t) = 50,$$

$$v_0 = 10,$$

$$x_0 = 20.$$

Solution - The velocity of the particle will be:

$$v(t) = 50t + C_1.$$

We have

$$v(0) = 50(0) + C_1 = 10,$$

so,

$$C_1 = 10.$$

The position of the particle will be:

$$x(t) = 25t^2 + 10t + C_2.$$

We have

$$x(0) = 25(0^2) + 10(0) + C_2 = 20,$$

so,

$$C_2 = 20.$$

The final position function will be:

$$x(t) = 25t^2 + 10t + 20.$$

1.2.15 Find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

$$a(t) = 4(t+3)^2,$$

$$v_0 = -1,$$

$$x_0 = 1.$$

Solution - If we take the antiderivative of the acceleration function we get:

$$v(t) = \frac{4}{3}(t+3)^3 + C_1.$$

Using the initial condition $v(0) = v_0 = -1$ we get:

$$\frac{4}{3}(0+3)^3 + C_1 = -1,$$

so,

$$C_1 = -37.$$

The velocity function will be:

$$v(t) = \frac{4}{3}(t+3)^3 - 37.$$

Taking the antiderivative of the velocity function we get:

$$x(t) = \frac{(t+3)^4}{3} - 37t + C_2.$$

Pluggin in the initial position $x(0) = x_0 = 1$ we get:

$$x(0) = \frac{(0+3)^4}{3} - 37(0) + C_2 = 1,$$

so,

$$C_2 = -26.$$

Our final position function is:

$$x(t) = \frac{(t+3)^4}{3} - 37t - 26.$$

1.2.27 A ball is thrown straight downward from the top of a tall building. The initial speed of the ball is $10m/s$. It strikes the ground with a speed of $60m/s$. How tall is the building?

Solution - I'll assume that the acceleration due to gravity is $10\frac{m}{s^2}$. The equation for the ball's velocity in the downward direction is:

$$v(t) = at + v_0.$$

Here $v_0 = 10\frac{m}{s}, a = 10\frac{m}{s^2}$, and $v(t_f) = 60\frac{m}{s}$. Solving this for t_f we get:

$$t_f = \frac{\left(60\frac{m}{s} - 10\frac{m}{s}\right)}{10\frac{m}{s^2}} = 5s.$$

The total distance traveled will be:

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0.$$

Here $x_0 = 0$ and we get:

$$x(t_f) = \frac{1}{2}\left(10\frac{m}{s^2}\right)(5s)^2 + \left(10\frac{m}{s}\right)(5s) = 125m + 50m = 175m.$$

1.2.35 A stone is dropped from rest at an initial height h above the surface of the earth. Show that the speed with which it strikes the ground is $v = \sqrt{2gh}$.

Solution - We could prove this using the conservation of energy. However, we should probably prove it using our distance equation. If $v_0 = x_0 = 0$ then the total distance the stone travels is:

$$h = \frac{1}{2}gt_f^2.$$

Its final velocity will be:

$$v(t_f) = gt_f = g\sqrt{\frac{2h}{g}} = \sqrt{2gh}.$$

1.2.43 Arthur Clark's *The Wind from the Sun* (1963) describes Diana, a spacecraft propelled by the solar wind. Its aluminized sail provides it with a constant acceleration of $0.001g = 0.0098m/s^2$. Suppose this spacecraft starts from rest at time $t = 0$ and simultaneously fires a projectile (straight ahead in the same direction) that travels at one-tenth of the speed $c = 3 \times 10^8 m/s$ of light. How long will it take the spacecraft to catch up with the projectile, and how far will it have traveled by then?

Solution - Set t_f to be the time at which the spacecraft catches up with the projectile. The total distance traveled by the spaceship will be:

$$x_1(t_f) = \frac{1}{2}at_f^2.$$

The total distance traveled by the projectile will be:

$$x_2(t_f) = vt_f.$$

When the spaceship catches up with the projectile, the distances are the same, and we have:

$$vt_f = \frac{1}{2}at_f^2.$$

Solving this for t_f we get:

$$t_f = \frac{2v}{a} \approx 6.122 \times 10^9 s.$$

So, about 194 years! The total distance traveled in that time will be:

$$vt_f \approx 1.837 \times 10^{17} m.$$

This is about 19.4 light years.[1]

[1] This is a very, very large distance. However, almost all the stars you see in the night sky are much, much farther away than this!

1.3.1 and 1.3.6 See Below

In Problems 1 through 10, we have provided the slope field of the indicated differential equation, together with one or more solution curves. Sketch likely solution curves through the additional points marked in each slope field.

1. $\dfrac{dy}{dx} = -y - \sin x$

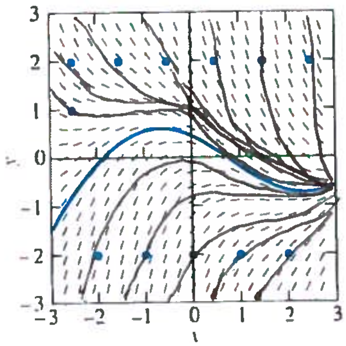


FIGURE 1.3.15.

2. $\dfrac{dy}{dx} = x + y$

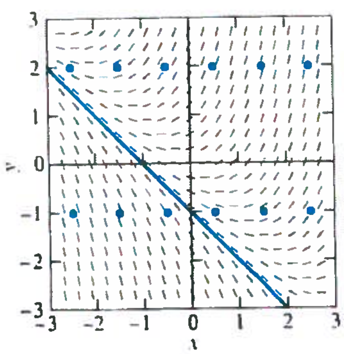


FIGURE 1.3.16.

3. $\dfrac{dy}{dx} = y - \sin x$

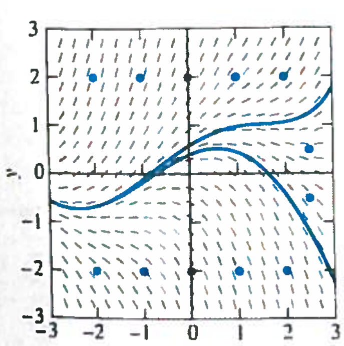


FIGURE 1.3.17.

4. $\dfrac{dy}{dx} = x - y$

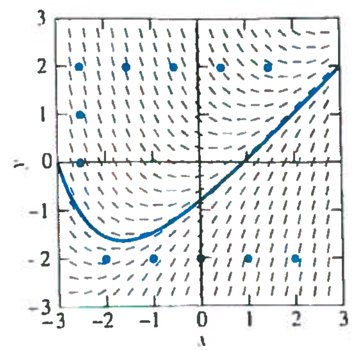


FIGURE 1.3.18.

5. $\dfrac{dy}{dx} = y - x + 1$

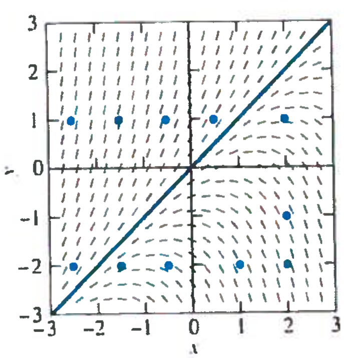


FIGURE 1.3.19.

6. $\dfrac{dy}{dx} = x - y + 1$

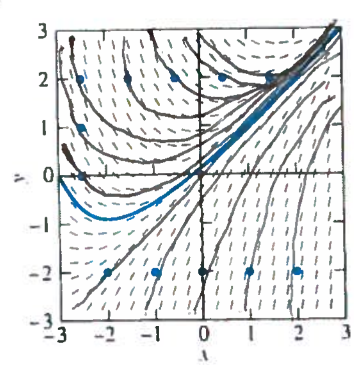


FIGURE 1.3.20.

1.3.9 See Below

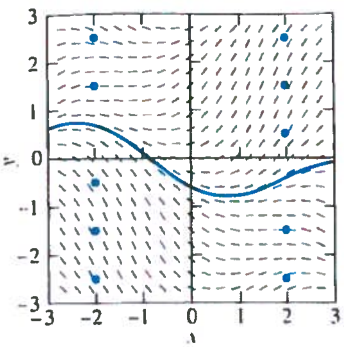


FIGURE 1.3.21.

8. $\dfrac{dy}{dx} = x^2 - y$

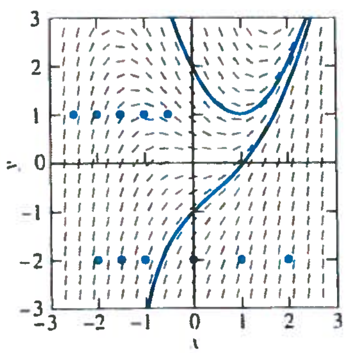


FIGURE 1.3.22.

9. $\dfrac{dy}{dx} = x^2 - y - 2$

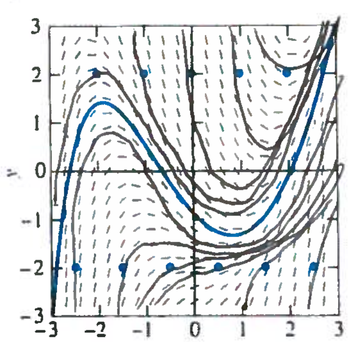


FIGURE 1.3.23.

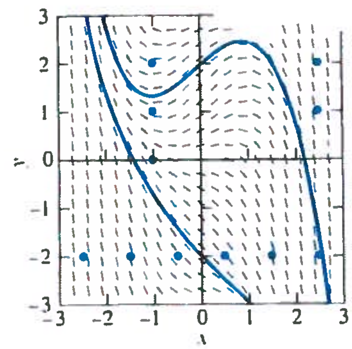


FIGURE 1.3.24.

In Problems 11 through 20, determine whether Theorem 1 does or does not guarantee existence of a solution of the given initial value problem. If existence is guaranteed, determine whether Theorem 1 does or does not guarantee uniqueness of that solution.

11. $\dfrac{dy}{dx} = 2x^2y^2; \quad y(1) = -1$

12. $\dfrac{dy}{dx} = x \ln y; \quad y(1) = 1$

13. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 1$

14. $\dfrac{dy}{dx} = \sqrt[3]{y}; \quad y(0) = 0$

15. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 2$

16. $\dfrac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 1$

17. $y\dfrac{dy}{dx} = x - 1; \quad y(0) = 1$

18. $y\dfrac{dy}{dx} = x - 1; \quad y(1) = 0$

19. $\dfrac{dy}{dx} = \ln(1 + y^2); \quad y(0) = 0$

20. $\dfrac{dy}{dx} = x^2 - y^2; \quad y(0) = 1$

In Problems 21 and 22, first use the method of Example 2 to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

21. $y' = x + y, \quad y(0) = 0; \quad y(-4) = ?$

22. $y' = y - x, \quad y(4) = 0; \quad y(-4) = ?$

1.3.11 Determine whether Theorem 1 does or does not guarantee existence of a solution of the given initial value problem. If existence is guaranteed, determine whether Theorem 1 does or does not guarantee uniqueness of that solution.

$$\frac{dy}{dx} = 2x^2y^2 \qquad\qquad y(1) = -1.$$

Solution - The function $f(x, y)$ is:

$$f(x, y) = 2x^2y^2.$$

Its partial derivative with respect to y is:

$$\frac{\partial f}{\partial y} = 4x^2y.$$

Both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous everywhere. So, there exists a unique solution around any initial value (a, b), including $(1, -1)$.

1.3.15 Determine whether Theorem 1 does or does not guarantee existence of a solution of the given initial value problem. If existence is guaranteed, determine whether Theorem 1 does or does not guarantee uniqueness of that solution.

$$\frac{dy}{dx} = \sqrt{x - y} \qquad\qquad y(2) = 2.$$

Solution - The function $f(x, y)$ is:

$$f(x, y) = \sqrt{x - y}.$$

Its partial derivative with respect to y is:

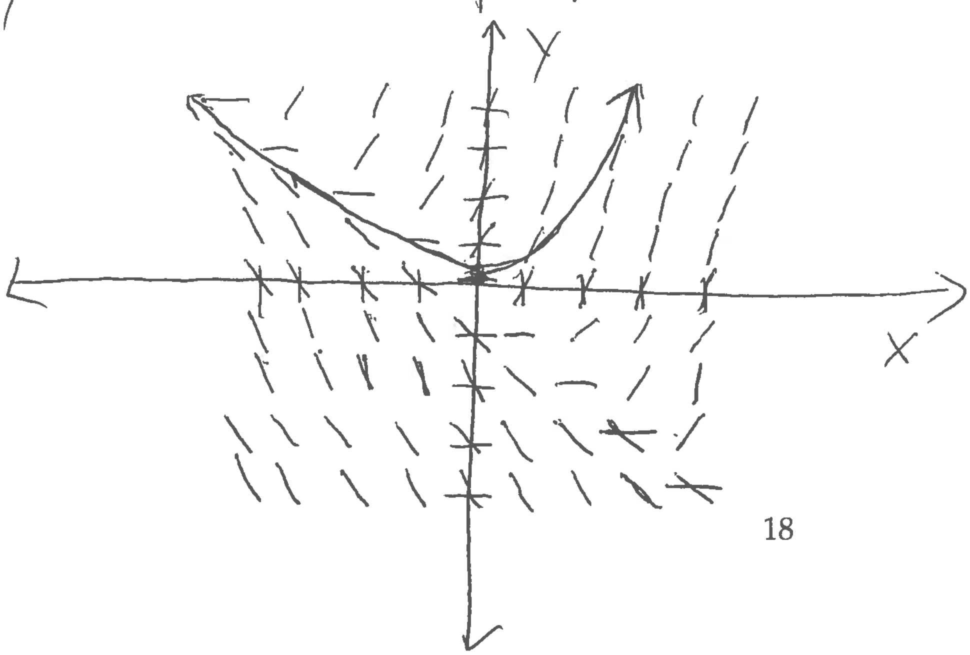
$$\frac{\partial f}{\partial y} = -\frac{1}{2\sqrt{x - y}}.$$

Both functions are continuous for $x > y$. At $(2, 2)$ we have $x = y$, and the function $\frac{\partial f}{\partial y}$ is undefined. So, existence is not guaranteed.

1.3.21 First use the method of Example 2 from the textbook to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution $y(x)$.

$$y' = x + y, \qquad y(0) = 0; \qquad y(-4) = ?$$

X+Y	-4	-3	-2	-1	0	1	2	3	4	X
4	0	1	2	3	4	5	6	7	8	
3	-1	0	1	2	3	4	5	6	7	
2	-2	-1	0	1	2	3	4	5	6	
1	-3	-2	-1	0	1	2	3	4	5	
0	-4	-3	-2	-1	0	1	2	3	4	
-1	-5	-4	-3	-2	-1	0	1	2	3	
-2	-6	-5	-4	-3	-2	-1	0	1	2	
-3	-7	-6	-5	-4	-3	-2	-1	0	1	
-4	-8	-7	-6	-5	-4	-3	-2	-1	0	
Y										



$y(-4) \approx 3$.

More room for Problem 1.3.21, if you need it.

1.3.29 Verify that if c is a constant, then the function defined piecewise by

$$y(x) = \begin{cases} 0 & x \leq c, \\ (x-c)^3 & x > c \end{cases}$$

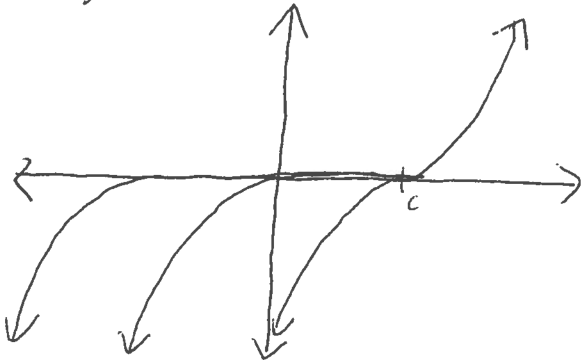
satisfies the differential equation $y' = 3y^{\frac{2}{3}}$ for all x. Can you also use the "left half" of the cubic $y = (x-c)^3$ in piecing together a solution curve of the differential equation? Sketch a variety of such solution curves. Is there a point (a, b) of the xy-plane such that the initial value problem $y' = 3y^{\frac{2}{3}}, y(a) = b$ has either no solution or a unique solution that is defined for all x? Reconcile your answer with Theorem 1.

$$y' = \begin{cases} 0 & x \leq c \\ 3(x-c)^2 & x > c \end{cases}$$

$$3y^{4/3} = \begin{cases} 0 & x \leq c \\ 3(x-c)^2 & x > c \end{cases}$$

So, it checks out.

You can also use the left half to piece together solution curves.



The function $f(x, y)$ is:

$$f(x, y) = 3y^{\frac{2}{3}}.$$

Its partial derivative with respect to y is:

$$\frac{\partial f}{\partial y} = \frac{2}{y^{\frac{1}{3}}}.$$

The function $f(x, y)$ is continuous everywhere. The function $\frac{\partial f}{\partial y}$ is continuous for $y \neq 0$. So, we're guaranteed a local unique solution if $y \neq 0$. For any (a, b) with $b \neq 0$ we're guaranteed a local unique solution.

There are *no* initial conditions for which there is a unique global solution. Theorem 1 only guarantees us a unique local solution.

Section 1.4 - Separable Equations and Applications

1.4.1 Find the general solution (implicit if necessary, explicit if convenient) to the differential equation

$$\frac{dy}{dx} + 2xy = 0$$

Solution - We can rewrite the differential equation as:

$$\frac{dy}{dx} = -2xy.$$

This is a separable equation, and so after a little algebra we can write it as:

$$\frac{dy}{y} = -2xdx.$$

Taking the antiderivative of both sides we get:

$$\int \frac{dy}{y} = -2\int xdx,$$

$$\Rightarrow \ln y = -x^2 + C.$$

Exponentiating both sides, and being a little sloppy with the unknown constant C,[2] we get:

$$y = Ce^{-x^2}.$$

[2] As always.

1.4.3 Find the general solution (implicit if necessary, explicit if convenient) to the differential equation

$$\frac{dy}{dx} = y \sin x$$

Solution - This is a separable equation, and we can rewrite it as:

$$\frac{dy}{y} = \sin x dx.$$

Taking the antiderivative of both sides we get:

$$\int \frac{dy}{y} = \int \sin x dx,$$
$$\Rightarrow \ln y = -\cos x + C.$$

Exponentiating both sides we get:

$$y = Ce^{-\cos x}.$$

1.4.17 Find the general solution (implicit if necessary, explicit if convenient) to the differential equation

$$y' = 1 + x + y + xy.$$

Primes denote the derivatives with respect to x. (Suggestion: Factor the right-hand side.)

Solution - We can factor the right-hand side of the equation above to get:

$$y' = (1 + x)(1 + y).$$

In this form it is obviously a separable differential equation, and we can rewrite it as:

$$\frac{dy}{1 + y} = (1 + x)dx.$$

If we take the antiderivative of both side we get:

$$\int \frac{dy}{1 + y} = \int (1 + x)dx,$$

$$\Rightarrow \ln(1 + y) = \frac{x^2}{2} + x + C.$$

Exponentiating both sides we get:

$$y + 1 = Ce^{\frac{x^2}{2} + x}.$$

So, our final solution is:

$$y = Ce^{\frac{x^2}{2} + x} - 1.$$

1.4.19 Find the explicit particular solution to the initial value problem

$$\frac{dy}{dx} = ye^x, \qquad y(0) = 2e.$$

Solution - This is a separable equation, and we can rewrite it as:

$$\frac{dy}{y} = e^x dx.$$

Taking the antiderivative of both sides we get:

$$\int \frac{dy}{y} = \int e^x dx,$$
$$\Rightarrow \ln y = e^x + C.$$

Exponentiating both sides we get:

$$y = Ce^{e^x}.$$

If we plug in $y(0) = 2e$ and solve for C we get:

$$2e = y(0) = Ce^{e^0} = Ce^1 = Ce.$$

So, we can see $C = 2$, and the solution to the initial value problem is:

$$y = 2e^{e^x}.$$

1.4.31 Discuss the difference between the differential equations $(dy/dx)^2 = 4y$ and $dy/dx = 2\sqrt{y}$. Do they have the same solution curves? Why or why not? Determine the points (a, b) in the plane for which the initial value problem $y' = 2\sqrt{y}$, $y(a) = b$ has **(a)** no solution, **(b)** a unique solution, **(c)** infinitely many solutions.

Solution - The solution curves for $\frac{dy}{dx} = 2\sqrt{y}$ will be solution curves for $\left(\frac{dy}{dx}\right)^2 = 4y$. However, so will the solution curves for $\frac{dy}{dx} = -2\sqrt{y}$. So, no, they do not have the same solution curves.

The function $2\sqrt{y}$ is continuous for $y \geq 0$. Its partial derivative with respect to y, $\frac{1}{\sqrt{y}}$, is continuous for $y > 0$.

The differential equation $\frac{dy}{dx} = 2\sqrt{y}$ is separable, and we can solve it:

$$\int \frac{dy}{\sqrt{y}} = \int 2dx,$$

$$\Rightarrow 2\sqrt{y} = 2x + C.$$

Solving this for y we get:

$$y = (x + C)^2.$$

(a) - There will be *no solution* if $b < 0$.

(b) - There will be (locally) a unique solution for $b > 0$.

(c) - If $b = 0$ there will be infinitely many solutions. This is similar to the situation we saw with Problem 1.3.29.

1.4.35 (Radiocarbon dating) Carbon extracted from an ancient skull contained only one-sixth as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

Solution - We have the relation:

$$\frac{1}{6} = \left(\frac{1}{2}\right)^{\frac{t}{T_{\frac{1}{2}}}},$$

where $T_{\frac{1}{2}}$ is the half-life of ^{14}C, about 5,700 years.

If we solve this for t we get:

$$t = \frac{T_{\frac{1}{2}} \ln\left(\frac{1}{6}\right)}{\ln\left(\frac{1}{2}\right)} = 14,734 \text{ years.}$$

So, the skull is approximately 15,000 years old.

1.4.53 Thousands of years ago ancestors of the Native Americans crossed the Bering Strait from Asia and entered the western hemisphere. Since then, they have fanned out across North and South America. The single language that the original Native Americans spoke has since split into many Indian "language families." Assume that the number of these language families has been multiplied by 1.5 every 6000 years. There are now 150 Native American language families in the western hemisphere. About when did the ancestors of today's Native Americans arrive?

Solution - We have the relation:

$$150 = (1.5)^{\frac{t}{6000}}.$$

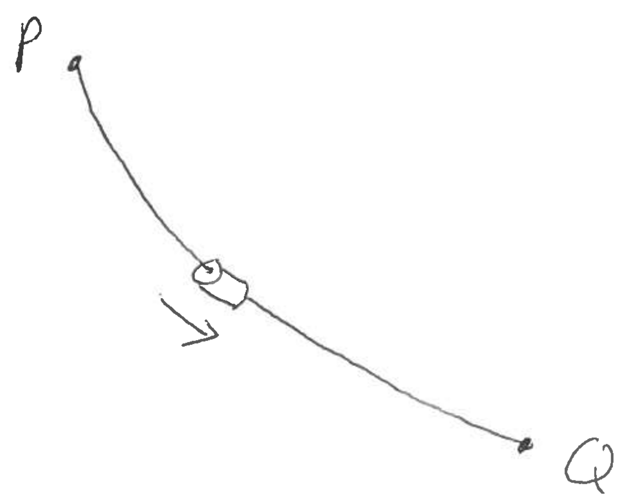
If we solve this for t we get:

$$t = 6000 \left(\frac{\ln 150}{\ln 1.5} \right) \approx 74,000 \text{ years ago.}$$

1.4.68 The figure below shows a bead sliding down a frictionless wire from point P to point Q. The *brachistochrone problem* asks what shape the wire should be in order to minimize the bead's time of descent from P to Q. In June of 1696, John Bernoulli proposed this problem as a public challenge, with a 6-month deadline (later extended to Easter 1697 at George Leibniz's request). Isaac Newton, then retired from academic life and serving as Warden of the Mint in London, received Bernoulli's challenge on January 29, 1697. The very next day he communicated his own solution - the curve of minimal descent time is an arc of an inverted cycloid - to the Royal Society of London. For a modern derivation of this result, suppose the bead starts from rest at the origin P and let $y = y(x)$ be the equation of the desired curve in a coordinate system with the y-axis pointing downward. Then a mechanical analogue of Snell's law in optics implies that

$$\frac{\sin\alpha}{v} = constant$$

where α denotes the angle of deflection (from the vertical) of the tangent line to the curve - so $\cot\alpha = y'(x)$ (why?) - and $v = \sqrt{2gy}$ is the bead's velocity when it has descended a distance y vertically (from $KE = \frac{1}{2}mv^2 = mgy = -PE$).



(a) First derive from $\sin\alpha/v = constant$ the differential equation

$$\frac{dy}{dx} = \sqrt{\frac{2a-y}{y}}$$

where a is an appropriate positive constant.

Solution - Denote the constant $\frac{\sin\alpha}{v}$ with the letter C. Then, as

$$y'(x) = \cos\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{\cos\alpha}{vC},$$

and

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-v^2C^2},$$

we have:

$$\frac{dy}{dx} = \frac{\sqrt{1-v^2C^2}}{vC}.$$

Now, $v = \sqrt{2gy}$, so this equals:

$$\frac{dy}{dx} = \frac{\sqrt{1-2C^2gy}}{C\sqrt{2gy}} = \frac{\sqrt{1-2C^2gy}}{\sqrt{2C^2gy}} = \sqrt{\frac{2a-y}{y}},$$

with $a = \dfrac{1}{4C^2g}$.

(b) Substitute $y = 2a\sin^2 t, dy = 4a \sin t \cos t dt$ in the above differential equation to derive the solution

$$x = a(2t - \sin 2t), \qquad y = a(1 - \cos 2t)$$

for which $t = y = 0$ when $x = 0$. Finally, the substitution of $\theta = 2a$ in the equations for x and y yields the standard parametric equations $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ of the cycloid that is generated by a point on the rim of a circular wheel of radius a as it rolls along the x-axis.

Solution - The equation $\dfrac{dy}{dx} = \sqrt{\dfrac{2a - y}{y}}$ becomes

$$\frac{4a\sin t\cos t dt}{dx} = \sqrt{\frac{2a - 2a\sin^2 t}{2a\sin^2 t}},$$

$$\Rightarrow 4a\sin t\cos t\left(\frac{dt}{dx}\right) = \sqrt{\frac{1 - \sin^2 t}{\sin^2 t}} = \frac{cost}{\sin t} = \cot t.$$

From this we get:

$$4a\sin^2 t\left(\frac{dt}{dx}\right) = 1,$$

$$\Rightarrow 4a\sin^2 t = \frac{dx}{dt}.$$

If we use the trigonometric identity $\sin^2 t = \frac{1-\cos 2t}{2}$ this equation becomes:

$$2a(1 - \cos(2t)) = \frac{dx}{dt}.$$

Integrating this we get:

$$x(t) = a(2t - \sin(2t)) + C.$$

If $x(0) = 0$ this implies $C = 0$, so

$$x(t) = a(2t - \sin(st)).$$

On the other hand, we have

$$y(t) = 2a\sin^2 t = 2a\left(\frac{1 - \cos(2t)}{2}\right) = a(1 - \cos(2t)).$$

So, combining what we've derived we get:

$$x(t) = a(2t - \sin(2t)) \qquad\qquad y(t) = a(1 - \cos(2t)).$$

Q.E.D.