Math 2280 - Assignment 11

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Section 8.1 - 2, 8, 13, 21, 25

Section 8.2 - 1, 7, 14, 17, 32

Section 8.3 - 1, 8, 15, 18, 24

Section 8.1 - Introduction and Review of Power Series

8.1.2 - Find the power series solution to the differential equation

$$y' = 4y,$$

and determine the radius of convergence for the series. Also, identify the series solution in terms of familiar elementary functions.

Solution - We set up the power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n,$$

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

If we plug these into our differential equation we get:

$$\sum_{n=1}^{\infty} n c_n x^{n-1} - 4\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+1)c_{n+1} - 4c_n] x^n = 0.$$

Using the identity principle from this we get the recursion relation:

$$c_{n+1} = \frac{4c_n}{n+1}.$$

The first few terms are

$$c_0 = c_0,$$

$$c_1 = \frac{4c_0}{1},$$

$$c_2 = \frac{4c_1}{2} = \frac{4^2 c_0}{2 \times 1},$$

$$c_3 = \frac{4c_2}{3} = \frac{4^3 c_0}{3 \times 2 \times 1},$$

and in general,

$$c_n = \frac{4^n c_0}{n!}.$$

The radius of convergence for our series is

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \frac{\frac{4^n}{n!}}{\frac{4^{n+1}}{(n+1)!}} = \lim_{n\to\infty} \frac{n+1}{4} = \infty.$$

So, the power series converges for all x, and the solution to this differential equation is:

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{4^n x^n}{n!} = c_0 \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = c_0 e^{4x}.$$

But, we already knew that, didn't we! :)

8.1.8 - Find the power series solution to the differential equation

$$2(x+1)y' = y,$$

and determine the radius of convergence for the series. Also, identify the series solution in terms of familiar elementary functions.

Solution - We set up the power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n,$$

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}.$$

If we plug these into our differential equation we get:

$$\sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=1}^{\infty} 2nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} (2(n+1)c_{n+1} - c_n)x^n = 0.$$

Using the identity principle the x^0 term gives us:

$$2c_1 - c_0 = 0 \Rightarrow c_1 = \frac{c_0}{2}.$$

The higher order terms give us:

$$c_{n+1} = \frac{(1-2n)c_n}{2(n+1)}.$$

The first few terms are

$$c_0 = c_0,$$

$$c_1 = \frac{c_0}{2},$$

$$c_2 = -\frac{c_1}{2 \cdot 2} = -\frac{c_0}{2^2 \cdot 2!},$$

$$c_3 = -\frac{3c_2}{2 \cdot 3} = \frac{3c_0}{2^3 \cdot 3!},$$

$$c_4 = -\frac{5c_3}{2(4)} = -\frac{15c_0}{2^4 \cdot 4!},$$

and in general,

$$c_n = \frac{(-1)^{n+1}(2n-3)!!c_0}{2^n n!},$$

for $n \geq 2$. Here $(2n-3)!!$ means the product of all the odd integers up to $(2n-3)$. The radius of convergence for our series is

$$\lim_{n\to\infty} \left\| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \frac{\frac{c_0(2n-3)!!(-1)^{n+1}}{2^n n!}}{\frac{c_0(2n-1)!!(-1)^{n+2}}{2^{n+1}(n+1)!}} = \lim_{n\to\infty} \frac{2(n+1)}{2n-1} = 1.$$

So, the power series converges for all $|x| < 1$, and the solution to this differential equation is:

$$y(x) = c_0 \left(1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2n-3)!!}{2^n n!} x^n \right).$$

Now, the Maclaurin series expansion for $\sqrt{1+x}$ is:

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \cdots .$$

This is just our series above, and so we have that our solution is

$$y(x) = c_0\sqrt{1+x}.$$

8.1.13 - Find two linearly independent power series solutions to the differential equation

$$y'' + 9y = 0,$$

and determine the radius of convergence for each series. Also, identify the general solution in terms of familiar elementary functions.

Solution - We set up the power series:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n,$$

$$y'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1},$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}.$$

If we plug these into our differential equation we get:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} 9 c_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + 9 c_n] x^n = 0.$$

Applying the identity principle, the recurrence relation our coefficients must satisfy is:

$$c_{n+2} = \frac{-9c_n}{(n+2)(n+1)}.$$

This splits our series into even and odd terms. For the even terms we have:

$$c_0 = c_0,$$

$$c_2 = \frac{-9c_0}{2 \times 1},$$

$$c_4 = \frac{-9c_2}{4 \times 3} = \frac{9^2 c_0}{4 \times 3 \times 2 \times 1},$$

and in general,

$$c_{2n} = \frac{(-1)^n 3^{2n} c_0}{(2n)!}.$$

Using the same reasoning for the odd terms we get the general formula:

$$c_{2n+1} = \frac{(-1)^n 3^{2n} c_1}{(2n+1)!}.$$

The radius of convergence for either of our series will be:

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)}{3k_n} \right| = \infty,$$

$$\text{where } k_n = \begin{cases} \frac{c_0}{c_1} & n\ odd \\ \frac{c_1}{c_0} & n\ even \end{cases}$$

So, the two power series solutions converge for all x. The solutions are:

$$
\begin{aligned}
y(x) &= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!} + \frac{c_1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} \\
&= C_1 \cos(3x) + C_2 \sin(3x),
\end{aligned}
$$

where C_1, C_2 are arbitrary constants.

8.1.21 - For the initial value problem

$$y'' - 2y' + y = 0;$$

$$y(0) = 0, y'(0) = 1,$$

derive a recurrence relation giving c_n for $n \geq 2$ in terms of c_0 or c_1 (or both). Then apply the given initial conditions to find the values of c_0 and c_1. Next, determine c_n and, finally, identify the particular solution in terms of familiar elementary functions.

Solution - Plugging in a power series solution into the ODE we get:

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=0}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

From this we get that c_0 and c_1 are arbitrary, and the rest of the coefficients must satisfy the relation:

$$c_{n+2}(n+2)(n+1) - 2c_{n+1}(n+1) + c_n = 0$$

$$\Rightarrow c_{n+2} = \frac{2c_{n+1}(n+1) - c_n}{(n+2)(n+1)}.$$

Now, $y(0) = c_0 = 0$, $y'(0) = c_1 = 1$, and the higher-order terms are:

$$c_2 = \frac{2}{2} = 1,$$

$$c_3 = \frac{2(1)(2) - 1}{(3)(2)} = \frac{1}{2} = \frac{1}{2!},$$

$$c_4 = \frac{2\left(\frac{1}{2}\right)(3) - 1}{(4)(3)} = \frac{1}{6} = \frac{1}{3!},$$

$$c_5 = \frac{2\left(\frac{1}{6}\right)(4) - \frac{1}{2}}{(5)(4)} = \frac{1}{24} = \frac{1}{4!},$$

and in general,

$$c_n = \frac{1}{(n-1)!}.$$

So,

$$y(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = xe^x.$$

8.1.25 - For the initial value problem

$$y'' = y' + y;$$
$$y(0) = 0, \quad y'(0) = 1,$$

derive the power series solution

$$y(x) = \sum_{n=1}^{\infty} \frac{F_n}{n!} x^n$$

where $\{F_n\}_{n=0}^{\infty}$ is the sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ of *Fibonacci numbers* defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-2} + F_{n-1}$ for $n > 1$.

Solution - If we plug a power series solution into the given differential equation we get:

$$\sum_{n=2}^{\infty} n(n+1)c_n x^{n-2} = \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n.$$

The constants c_0 and c_1 are determined by the initial conditions $y(0) = c_0 = 0$, and $y'(0) = c_1 = 1$. For $n \geq 2$ if we apply the identity principle to our power series ODE above we get the recurrence relation:

$$c_{n+2} = \frac{c_{n+1}(n+1) + c_n}{(n+2)(n+1)}.$$

Assume the coefficients in the series are of the form $c_n = \frac{F_n}{n!}$ up to $n+1$. Then

$$c_{n+1} = \frac{\frac{F_{n+1}(n+1)}{(n+1)!} + \frac{F_n}{n!}}{(n+2)(n+1)} = \frac{F_{n+1} + F_n}{(n+2)!} = \frac{F_{n+2}}{(n+2)!}.$$

So, by induction, we get

$$c_n = \frac{F_n}{n!},$$

and

$$y(x) = \sum_{n=1}^{\infty} \frac{F_n}{n!} x^n.$$

Section 8.2 - Series Solutions Near Ordinary Points

8.2.1 - Find a general solution in powers of x to the differential equation

$$(x^2-1)y'' + 4xy' + 2y = 0.$$

State the recurrence relation and the guaranteed radius of convergence.

Solution - A power series solution $y(x)$ and its derivatives will have the forms:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n;$$

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1};$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

If we plug these into the ODE we get:

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n - \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + 4\sum_{n=0}^{\infty} n c_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0.$$

This simplifies to:

$$\sum_{n=0}^{\infty} [(n(n-1)+4n+2)c_n - (n+2)(n+1)c_{n+2}]x^n = 0.$$

From this we get the recurrence relation:

$$c_{n+2} = \frac{(n^2+3n+2)}{(n+2)(n+1)}c_n = \frac{(n+2)(n+1)}{(n+2)(n+1)}c_n = c_n.$$

So, we specify c_0 and c_1, and the rest of the coefficients are determined by the above recurrence relation. This gives us a geometric series, and the corresponding solution to the ODE is:

$$y(x) = c_0 \sum_{n=0}^{\infty} x^{2n} + c_1 x \sum_{n=0}^{\infty} x^{2n} = \frac{c_0 + c_1 x}{1 - x^2}.$$

The radius of convergence here is $\rho = 1$, which is the distance from $x = 0$ to the closest root of $x^2 - 1$, which is where the singular points of our differential equation are.

8.2.7 - Find a general solution in powers of x to the differential equation

$$(x^2+3)y'' - 7xy' + 16y = 0.$$

State the recurrence relation and the guaranteed radius of convergence.

Solution - Plugging a power series solution into our differential equation we get:

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n + \sum_{n=0}^{\infty} 3n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} 7nc_n x^n + \sum_{n=0}^{\infty} 16c_n x^n = 0.$$

Simplifying this we get:

$$\sum_{n=0}^{\infty}[(n^2-8n+16)c_n + 3(n+2)(n+1)c_{n+2}]x^n = 0.$$

From this we get the recurrence relation:

$$c_{n+2} = -\frac{(n-4)^2}{3(n+2)(n+1)}c_n.$$

From this we can specify c_0 and c_1 arbitrarily, and the rest of the coefficients are determined by the above recurrence relation.

The even terms are easy:

$$c_0 = c_0; \quad c_2 = -\frac{16}{3(2)(1)}c_0 = -\frac{8}{3}c_0;$$

$$c_4 = -\frac{4c_2}{3(4\cdot 3)} = \frac{8}{27}c_0; \quad c_6 = 0;$$

and all higher-order even terms are 0.

The odd terms are a little more tricky:

$$c_1 = c_1; \quad c_3 = -\frac{9}{3(3\cdot 2)}c_1;$$

$$c_5 = -\frac{c_3}{3\cdot(5\cdot 4)} = \frac{9}{3^2(5\cdot 4\cdot 3\cdot 2\cdot 1)}c_1; \quad c_7 = -\frac{c_5}{3\cdot(7\cdot 6)} = -\frac{9}{3^3 7!};$$

$$c_9 = -\frac{9c_7}{3\cdot(9\cdot 8)} = \frac{81c_1}{3^4 9!};$$

and in general for $n \geq 3$:

$$c_{2n+1} = \frac{(-1)^n[(2n-5)!!]^2 c_1}{3^{n-2}(2n+1)!}c_1.^1$$

So, our solution is:

$$y(x) = c_0\left(1 - \frac{8}{3}x^2 + \frac{8}{27}x^4\right) + c_1\left(x - \frac{x^3}{2} + \frac{x^5}{120} + \sum_{n=3}^{\infty}\frac{(-1)^n[(2n-5)!!]^2 x^{2n+1}}{3^{n-2}(2n+1)!}\right).$$

The point $x = 0$ is $\sqrt{3}$ from the nearest root of x^2+3 (its roots are $\pm\sqrt{3}i$), so the guaranteed radius of convergence is $\sqrt{3}$.

[1] There is a typo in the answer in the back of the book here. The back of the book does not square $[(2n-5)!!]$.

8.2.14 - Find a general solution in powers of x to the differential equation

$$y'' + xy = 0.$$[2]

State the recurrence relation and the guaranteed radius of convergence.

Solution - Plugging a power series solution into the above ODE we get:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

The x^0 constant terms just shows up in the first sum for $n = 2$, so by the identity principle $2c_2 = 0 \Rightarrow c_2 = 0$.

The constants c_0, c_1 will be arbitrary, and after simplification our differential equation becomes:

$$\sum_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} + c_n]x^{n+1} = 0.$$

From this we get the recurrence relation:

$$c_{n+3} = -\frac{c_n}{(n+3)(n+2)}.$$

From this we get all the terms of the form c_{3n+2} will be 0, and $c_2 = 0$, while the other terms will be:

[2] An *Airy equation*.

$$c_{3n} = \frac{(-1)^n(1 \cdot 4 \cdot 7 \cdots (3n-2))}{(3n)!} c_0,$$

and similarly,

$$c_{3n+1} = \frac{(-1)^n(2 \cdot 5 \cdot 8 \cdots (3n-1))}{(3n+1)!} c_1.$$

From this we get the solution:

$$y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n(1 \cdot 4 \cdot 7 \cdots (3n-2))}{(3n)!} x^{3n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n(2 \cdot 5 \cdot 8 \cdots (3n-1))}{(3n+1)!} x^{3n+1}.$$

There are no singular points in our differential equation, so this series is guaranteed to converge everywhere.

8.2.17 - Use power series to solve the initial value problem

$$y'' + xy' - 2y = 0;$$

$$y(0) = 1, \quad y'(0) = 0.$$

Solution - If we plug in a power series solution into the differential equation we get:

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} nc_n x^n - 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + (n-2)c_n]x^n = 0.$$

This gives us the recurrence relation:

$$c_{n+2} = -\frac{(n-2)c_n}{(n+2)(n+1)}.$$

The even terms will be $c_0 = c_0$, $c_2 = \frac{2c_0}{2 \cdot 1} = c_0$, $c_4 = 0$, and all other even terms are 0.

The odd terms will be:

$$c_1 = c_1;$$

$$c_3 = \frac{c_1}{3 \cdot 2} = \frac{c_1}{3!};$$

$$c_5 = -\frac{c_3}{5 \cdot 4} = -\frac{c_1}{5!};$$

and in general:

$$c_{2n+1} = \frac{(-1)^{n+1}(2n-3)!!}{(2n+1)!}x^{2n+1}.$$

So, the general solution is:

$$y(x) = c_0(1+x^2) + c_1\left(x + \frac{x^3}{3!} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(2n-3)!!}{(2n+1)!}x^{2n+1}\right).$$

Now, $y(0) = c_0 = 1$, and $y'(0) = c_1 = 0$. So, the solution to the IVP is:

$$y(x) = 1 + x^2.$$

8.2.32 - Follow the steps outlined in this problem to establish *Rodrigues's formula*

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

for the nth-degree Legendre polynomial.

(a) Show that $v = (x^2-1)^n$ satisfies the differential equation

$$(1-x^2)v' + 2nxv = 0.$$

Differentiate each side of this equation to obtain

$$(1-x^2)v'' + 2(n-1)xv' + 2nv = 0.$$

(b) Differentiate each side of the last equation n times in succession to obtain

$$(1-x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)} = 0.$$

Thus $u = v^{(n)} = D^n(x^2-1)^n$ satisfies Legendre's equation of order n.

(c) Show that the coefficient of x^n in u is $(2n)!/n!$; then state why this proves Rodrigues' formula. (Note that the coefficient of x^n in $P_n(x)$ is $(2n)!/[2^n(n!)^2]$.)

Solution -

(a) - We know $v' = \dfrac{d}{dx}(x^2-1)^n = n(x^2-1)^{n-1}2x.$

So,

$$(1-x^2)v' + 2nxv = (1-x^2)n(x^2-1)^{n-1}2x + 2nx(x^2-1)^n = \\ -2nx(x^2-1)^n + 2nx(x^2-1)^n = 0.$$

So, it satisfies the ODE. If we differentiate the ODE we get:

$$(1-x^2)v'' - 2xv' + 2nxv' + 2nv = 0$$

$$\Rightarrow (1-x^2)v'' + 2(n-1)xv' + 2nv = 0.$$

(b) - We can prove this by induction. I will prove for $k \leq n$ the result of differentiating

$$(1-x^2)v'' + 2(n-1)xv' + 2nv = 0$$

k times is:

$$0 = (1-x^2)v^{(k+2)} + 2(n-(k+1))xv^{(k+1)} + \sum_{m=0}^{k} 2(n-m)v^{(k)}.$$

For the base case $k = 0$ we get:

$$(1-x^2)v'' + 2(n-1)xv' + 2nv = 0,$$

which is our original equation, so it checks out. Now, suppose it's true for up to $k-1$ We differentiate

$$(1-x^2)v^{(k+1)} + 2(n-k)xv^{(k)} + \sum_{m=0}^{k-1} 2(n-m)v^{(k-1)}$$

to get

$$(1-x^2)v^{(k+2)} - 2xv^{(k+1)} + 2(n-k)xv^{(k+1)} + 2(n-k)v^{(k)} + \\ \sum_{m=0}^{k-1} 2(n-m)v^{(k)}$$

$$= (1-x^2)v^{(k+2)} + 2(n-(k+1))xv^{(k+1)} + \sum_{m=0}^{k} 2(n-m)v^{(k)}.$$

So, the formula works. If we plug in $k = n$ we get:

$$(1-x^2)v^{(n+2)} + 2(n-(n+1))xv^{(n+1)} + \left[2n\sum_{m=0}^{n} 1 - 2\sum_{m=0}^{n} m\right] v^{(n)}$$

$$= (1-x^2)v^{(n+2)} - 2xv^{(n+1)} + \left(2n(n+1) - 2\left(\frac{n^2+n}{2}\right)\right) v^{(n)} =$$

$$(1-x^2)v^{(n+2)} - 2xv^{(n+1)} + (n^2+n)v^{(n)}$$

$$= (1-x^2)v^{(n+2)} - 2xv^{(n+1)} + n(n+1)v^{(n)}.$$

So, it works!

(c) Set

$$u = D^n(x^2-1)^n = \frac{d^n}{dx^n}(x^{2n} + \cdots) \quad \text{(where the dots represent lower order terms)}$$

$$2n(2n-1)(2n-2)\cdots(2n-(n-1))x^n + \cdots$$

$$= \frac{(2n)!}{n!}x^n + \cdots.$$

So, as $u = v^{(n)}$ satisfies Legendre's equation of order n, $\frac{u}{n!2^n}$ does as well. As explained in the textbook there is only one polynomial that satisfies Legendre's equation of order n, namely

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}.$$

So, it must be $kP_n = \dfrac{u}{n!2^n}$ where k is a constant. The highest order term in $P_n(x)$ is when $k = 0$ and is

$$P_n(x) = \frac{(2n)!}{2^n n!} x^n + \cdots.$$

Now,

$$\frac{u}{n!2^n} = \frac{(2n)!}{2^n (n!)^2} x^n + \cdots.$$

So, $k = 1$, and indeed:

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Section 8.3 - Regular Singular Points

8.3.1 - Determine whether $x = 0$ is an ordinary point, a regular singular point, or an irregular singular point for the differential equation

$$xy'' + (x - x^3)y' + (\sin x)y = 0.$$

If it is a regular singular point, find the exponents of the differential equation (the solutions to the indicial equation) at $x = 0$.

Solution - We can rewrite the differential equation as:

$$y'' + \frac{x - x^3}{x}y' + \frac{\sin(x)}{x}y = 0.$$

So,

$$P(x) = \frac{x - x^3}{x} = 1 - x^2$$
$$Q(x) = \frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots.$$

Both $P(x)$ and $Q(x)$ are analytic at $x = 0$. So, $x = 0$ is an ordinary point.

8.3.8 - Determine whether $x = 0$ is an ordinary point, a regular singular point, or an irregular singular point for the differential equation

$$(6x^2 + 2x^3)y'' + 21xy' + 9(x^2 - 1)y = 0.$$

If it is a regular singular point, find the exponents of the differential equation (the solutions to the indicial equation) at $x = 0$.

Solution - We rewrite the differential equation as:

$$y'' + \frac{21x}{6x^2 + 2x^3}y' + \frac{9(x^2 - 1)}{6x^2 + 2x^3}y = 0.$$

The coefficient functions are:

$$P(x) = \frac{21x}{6x^2 + 2x^3},$$

$$Q(x) = \frac{9(x^2 - 1)}{6x^2 + 2x^3}.$$

Both are singular at $x = 0$. Now,

$$p(x) = xP(x) = \frac{21}{6 + 2x},$$

$$q(x) = x^2Q(x) = \frac{9(x^2 - 1)}{6 + 2x}.$$

Both are analytic at $x = 0$. So, $x = 0$ is a regular singular point.

$$p(0) = p_0 = \frac{21}{6} = \frac{7}{2},$$

$$q(0) = q_0 = -\frac{3}{2}.$$

So, the indicial equation is:

$$r(r-1)+\frac{7}{2}r-\frac{3}{2}=0 \Rightarrow r^2+\frac{5}{2}r-\frac{3}{2}=(r+3)\left(r-\frac{1}{2}\right).$$

So, the exponents of the differential equation (the roots of the indicial equation) are $r=-3, \frac{1}{2}$.

8.3.15 - If $x = a \neq 0$ is a singular point of a second-order linear differential equation, then the substitution $t = x - a$ transforms it into a differential equation having $t = 0$ as a singular point. We then attribute to the original equation at $x = a$ the behavior of the new equation at $t = 0$. Classify (as regular or irregular) the singular points of the differential equation

$$(x-2)^2 y'' - (x^2 - 4)y' + (x+2)y = 0.$$

Solution - We can rewrite this differential equation as:

$$y'' - \frac{x^2 - 4}{(x-2)^2}y' + \frac{x+2}{(x-2)^2}y = 0$$

$$\Rightarrow y'' - \frac{x+2}{x-2}y' + \frac{x+2}{(x-2)^2}y = 0.$$

There is a singular point at $x = 2$. If we substitute $t = x - 2$ we get:

$$y'' - \frac{t+4}{t}y' + \frac{t+4}{t^2} = 0.$$

The point $t = 0$ is a singular point of this ODE. The functions:

$$p(t) = tP(t) = -(t+4);$$

$$q(t) = t^2 Q(t) = t + 4;$$

are both analytic at $t = 0$. So, $x = 2$ is a regular singular point.

8.3.18 - Find two linearly independent Frobenius series solutions (for $x > 0$) to the differential equation

$$2xy'' + 3y' - y = 0.$$

Solution - We rewrite the differential equation as:

$$y'' + \frac{3}{2x}y' - \frac{1}{2x}y = 0.$$

The coefficient functions are:

$$P(x) = \frac{3}{2x};$$
$$Q(x) = -\frac{1}{2x}.$$

So, $x = 0$ is a singular point. To check if it's a regular singular point we examine the funcitons:

$$p(x) = xP(x) = \frac{3}{2};$$
$$q(x) = x^2Q(x) = -\frac{x}{2}.$$

Both functions are analytic at $x = 0$, with leading terms $p(0) = p_0 = \frac{3}{2}$, and $q(0) = q_0 = 0$. The indicial equation is:

$$r(r-1) + \frac{3}{2}r = r^2 + \frac{1}{2}r = r\left(r + \frac{1}{2}\right).$$

The roots of the indicial equation are $r = 0, -\frac{1}{2}$. The Frobenius series solution will have the form:

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r};$$

$$y'(x) = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1};$$

$$y''(x) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}.$$

If we plug these into our differential equation we get:

$$\sum_{n=0}^{\infty} 2c_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} 3c_n (n+r) x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

The lowest order term in the series gives us the indicial equation. For the higher order terms we get:

$$\sum_{n=0}^{\infty} [(n+r+1)(2n+2n+3)c_{n+1} - c_n] x^{n+r} = 0.$$

The term c_0 is arbitrary, and the rest are determined by:

$$c_{n+1} = \frac{c_n}{(n+r+1)(2n+2r+3)}.$$

For $r = -\frac{1}{2}$ we get:

$$c_{n+1} = \frac{c_n}{(n+\frac{1}{2})(2n+2)} = \frac{c_n}{(2n+1)(n+1)},$$

and in general

$$c_{n+1} = \frac{c_0}{(n+1)!(2n+1)!!}.$$

For $r = 0$ we get:

$$c_{n+1} = \frac{c_n}{(n+1)(2n+3)},$$

and in general

$$c_{n+1} = \frac{c_0}{(n+1)!(2n+3)!!}.$$

So, our solution is:

$$y(x) = a_0 x^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!(2n+1)!!} + b_0 \left(1 + \sum_{n=1}^{\infty} \frac{x^n}{(n+1)!(2n+3)!!}\right).$$

8.3.24 - Find two linearly independent Frobenius series solutions (for $x > 0$) to the differential equation

$$3x^2y'' + 2xy' + x^2y = 0.$$

Solution - We can rewrite the differential equation as:

$$y'' + \frac{2}{3x}y' + \frac{1}{3}y = 0.$$

This gives us coefficient functions:

$$P(x) = \frac{2}{3x} \quad Q(x) = \frac{1}{3}.$$

The function $Q(x)$ is analytic at $x = 0$, but not $P(x)$, and therefore $x = 0$ is a singular point. However, the functions:

$$p(x) = xP(x) = \frac{2}{3} \quad q(x) = x^2Q(x) = \frac{x^2}{3},$$

are both analytic at $x = 0$, with leading terms $p(0) = p_0 = \frac{2}{3}$ and $q(0) = q_0 = 0$. The corresponding indicial equation is:

$$r(r-1) + \frac{2}{3}r = r^2 - \frac{1}{3}r = r(r - \frac{1}{3}).$$

So, the roots of the indicial equation are $r = 0, \frac{1}{3}$. A Frobenius series solution will have the form:

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r};$$

$$y'(x) = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1};$$

$$y''(x) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}.$$

If we plug these into our differential equation we get:

$$3\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r} + 2\sum_{n=0}^{\infty} c_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0.$$

The x^r term is:

$$3c_0 r(r-1) + 2c_0 r = c_0(3r^2 - r) = c_0 r(3r-1).$$

If r satisfies the indicial equation then this is automatically 0, so c_0 is arbitrary. The x^{r+1} term is:

$$3c_1(r+1)r + 2c_1(r+1) = c_1(3r^2 + 5r + 2).$$

The values $r = 0, \frac{1}{3}$ are not roots of the polynomial in r above, so for this term to be 0 we must have $c_1 = 0$.

For the higher order terms we get the recurrence relation:

$$[3(n+r+2)(n+r+1) + 2(n+r+2)]c_{n+2} + c_n = 0,$$

$$\Rightarrow c_{n+2} = -\frac{c_n}{(n+r+2)(3n+3r+5)}.$$

Now, all odd terms will be 0 as $c_1 = 0$. As for the even terms, for $r = 0$ we get:

$$a_0 = a_0,$$

$$a_2 = -\frac{a_0}{2 \cdot 5},$$

$$a_4 = -\frac{a_2}{4 \cdot 11} = \frac{a_0}{(2 \cdot 4)(5 \cdot 11)},$$

and in general,

$$a_{2n} = \frac{(-1)^n a_0}{2^n n!(5 \cdot 11 \cdots (6n-1))}.$$

For $r = \frac{1}{3}$ we get:

$$b_0 = b_0,$$

$$b_2 = -\frac{b_0}{7 \cdot 2},$$

$$b_4 = \frac{b_0}{(7 \cdot 13)(2 \cdot 4)},$$

$$b_6 = -\frac{b_0}{(7 \cdot 13 \cdot 17)(2 \cdot 4 \cdot 6)},$$

and in general,

$$b_{2n} = \frac{(-1)^n b_0}{2^n n!(7 \cdot 13 \cdot 17 \cdots (6n+1))}.$$

So, our solution is:

$$y(x) = a_0 x^{\frac{1}{3}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!(7 \cdot 13 \cdots (6n+1))} +$$
$$b_0 \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!(5 \cdot 11 \cdots (6n-1))} \right).$$