Math 2280 - Assignment 10

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Section 7.4 - 1, 5, 10, 19, 31

Section 7.5 - 1, 6, 15, 21, 26

Section 7.6 - 1, 6, 11, 14, 15

Section 7.4 - Derivatives, Integrals, and Products of Transforms

7.4.1 - Find the convolution $f(t) * g(t)$ of the functions

$$f(t) = t, \quad g(t) = 1.$$

Solution - Using the definition of convolution we get:

$$f(t) * g(t) = \int_0^t \tau d\tau = \left. \frac{\tau^2}{2} \right|_0^t = \frac{t^2}{2}.$$

7.4.5 - Find the convolution $f(t) * g(t)$ of the functions

$$f(t) = g(t) = e^{at}.$$

Solution -

$$f(t) * g(t) = \int_0^t e^{a\tau} e^{a(t-\tau)} d\tau = e^{at} \int_0^t d\tau = \tau e^{at}|_0^t = te^{at}.$$

7.4.10 - Apply the convolution theorem to find the inverse Laplace transform of the function

$$F(s) = \frac{1}{s^2(s^2+k^2)}.$$

Solution - We have

$$F(s) = \left(\frac{1}{s^2}\right)\left(\frac{1}{s^2+k^2}\right) = \mathcal{L}(t)\frac{1}{k}\mathcal{L}(\sin{(kt)}).$$

From this we get that our function $f(t)$ is the convolution:

$$f(t) = \frac{1}{k}(t * \sin{(kt)}) = \frac{1}{k}\int_0^t (t-\tau)\sin{(k\tau)}d\tau$$

$$= -\frac{1}{k}\int_0^t \tau\sin{(k\tau)}d\tau + \frac{t}{k}\int_0^t \sin{(k\tau)}d\tau$$

$$= \frac{1}{k}\left(\frac{\tau\cos{(k\tau)}}{k} - \frac{\sin{(k\tau)}}{k^2}\right)\Bigg|_0^t - \frac{t}{k^2}\cos{(k\tau)}\Bigg|_0^t$$

$$= \frac{t\cos{(kt)}}{k^2} - \frac{\sin{(kt)}}{k^3} - \frac{t\cos{(kt)}}{k^2} + \frac{t}{k^2}$$

$$= \frac{kt - \sin{(kt)}}{k^3}.$$

7.4.19 - Find the Laplace transform of the function

$$f(t) = \frac{\sin t}{t}.$$

Solution - We use the relation

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty f(\sigma)d\sigma,$$

where $F(\sigma) = \mathcal{L}(f(t))$.

For $f(t) = \sin(kt)$ we have

$$\mathcal{L}(\sin(t)) = \frac{1}{\sigma^2 + 1},$$

and therefore

$$\mathcal{L}\left(\frac{\sin(t)}{t}\right) = \int_s^\infty \frac{d\sigma}{\sigma^2 + 1} = \tan^{-1}(\sigma)|_s^\infty$$
$$= \tan^{-1}(\infty) - \tan^{-1}(s) = \frac{\pi}{2} - \tan^{-1}(s).$$

This is a correct and perfectly acceptable final answer, but using some trig identities we could also write it as:

$$\frac{\pi}{2} - \tan^{-1}(s) = \tan^{-1}\left(\frac{1}{s}\right).$$

7.4.31 - Transform the given differential equation to find a nontrivial solution such that $x(0) = 0$.

$$tx'' - (4t+1)x' + 2(2t+1)x = 0.$$

Solution - Using what we know about the Laplace transforms of derivatives we have:

$$\mathcal{L}(x'') = s^2X(s) - k, \quad \text{where } k = x'(0),$$

$$\mathcal{L}(x') = sX(s),$$

$$\mathcal{L}(x) = X(s).$$

Using these relations and Theorem 7.4.2 from the textbook we get

$$\mathcal{L}(tx'') = -\frac{d}{ds}(s^2X(s) - k) = -s^2X'(s) - 2sX(s),$$

$$\mathcal{L}(-tx') = sX'(s) + X(s),$$

$$\mathcal{L}(tx) = -X'(s).$$

So, the ODE becomes

$$(s^2 - 4s + 4)X'(s) + (3s - 6)X(s) = 0,$$

$$\Rightarrow (s-2)^2X'(s) + 3(s-2)X(s) = 0,$$

$$\Rightarrow X'(s) + \frac{3}{s-2}X(s) = 0.$$

We can rewrite this as:

$$\frac{X'(s)}{X(s)} = -\frac{3}{s-2}.$$

Integrating both sides we get:

$$\ln(X(s)) = -3\ln(s-2) + C$$
$$\Rightarrow X(s) = \frac{C}{(s-2)^3}.$$

The inverse Laplace transform of $X(s)$, our solution, is:

$$x(t) = Ce^{2t}t^2.$$

For this inverse Laplace transform we use the translation theorem and the relation $\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = t^2$.

Section 7.5 - Periodic and Piecewise Continuous Input Functions

7.5.1 - Find the inverse Laplace transform $f(t)$ of the function

$$F(s) = \frac{e^{-3s}}{s^2}.$$

Solution - Using the translation theorem

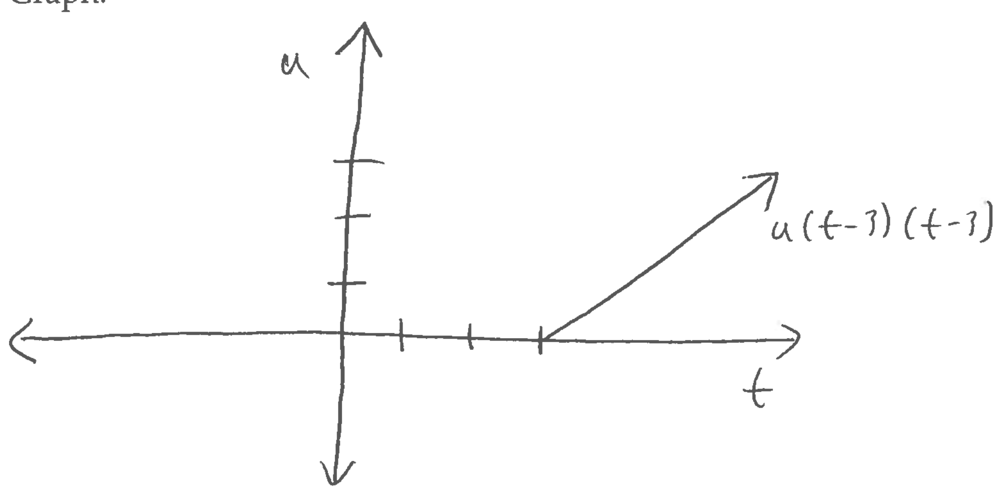
$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(e^{-3s}\left(\frac{1}{s^2}\right)\right) = u(t-3)f(t-3),$$

where $f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t.$

So,

$$f(t) = u(t-3)(t-3).$$

Graph:



7.5.6 - Find the inverse Laplace transform $f(t)$ of the function

$$F(s) = \frac{se^{-s}}{s^2 + \pi^2}.$$

Solution - Again applying the translation formula we have

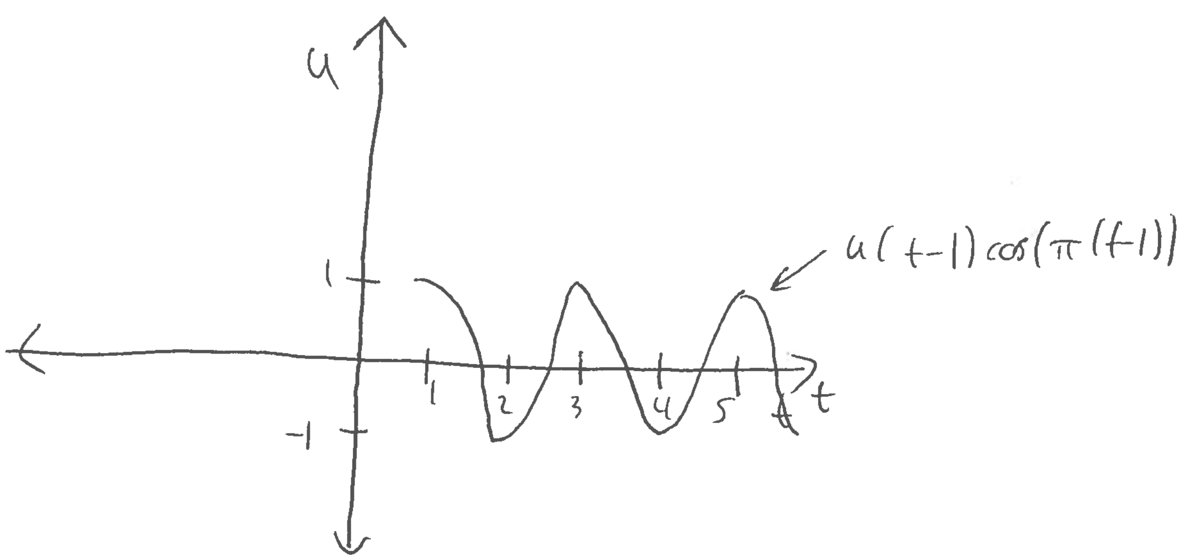
$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(e^{-s}\left(\frac{s}{s^2+\pi^2}\right)\right) = u(t-1)f(t-1),$$

where $f(t) = \mathcal{L}^{-1}\left(\frac{s}{s^2+\pi^2}\right) = \cos(\pi t)$.

So,

$$\mathcal{L}^{-1}(F(s)) = u(t-1)\cos(\pi(t-1)).$$

Graph:



7.5.15 - Find the Laplace transform of the function

$$f(t) = \sin t \text{ if } 0 \leq t \leq 3\pi;\ f(t) = 0 \text{ if } t > 3\pi.$$

Solution - We can write the function defined above using step functions as

$$\sin(t)(u(t) - u(t - 3\pi)).$$

If we use the identity $\sin(t) = -\sin(t - 3\pi)$ we can rewrite the above equation as

$$\sin(t)u(t) + \sin(t - 3\pi)u(t - 3\pi).$$

So, the Laplace transform will be:

$$\mathcal{L}(u(t)\sin(t) + u(t - 3\pi)\sin(t - 3\pi)) = \frac{1}{s^2 + 1} + \frac{e^{-3\pi s}}{s^2 + 1}$$

$$= \frac{1 + e^{-3\pi s}}{s^2 + 1}.$$

7.5.21 - Find the Laplace transform of the function

$$f(t) = t \text{ if } t \leq 1;\ f(t) = 2 - t \text{ if } 1 \leq t \leq 2;\ f(t) = 0 \text{ if } t > 2.$$

Solution - We can write the function as

$$f(t) = \begin{cases} t & t \leq 1 \\ 2 - t & 1 \leq t \leq 2 \\ 0 & t > 2 \end{cases}$$

Using step functions we can write this as:

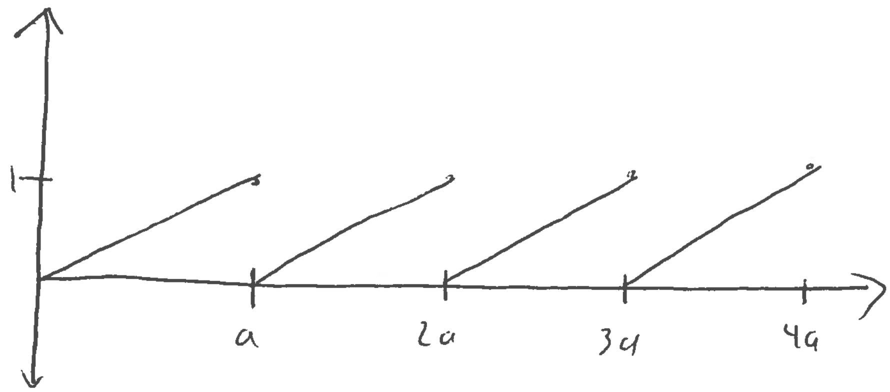
$$f(t) = t - u(t-1)t + u(t-1)(2-t) - u(t-2)(2-t)$$
$$= t - 2(t-1)u(t-1) + (t-2)u(t-2).$$

The Laplace transform of this function will be:

$$\mathcal{L}(f(t)) = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}$$
$$= \frac{(1 - e^{-s})^2}{s^2}.$$

7.5.26 - Apply Theorem 2 to show that the Laplace transform of the saw-tooth function $f(t)$ pictured below is

$$F(s) = \frac{1}{as^2} - \frac{e^{-as}}{s(1-e^{-as})}.$$



Solution - The period here is a. So,

$$\mathcal{L}(f(t)) = \frac{1}{1-e^{-as}}\int_0^a \frac{te^{-st}}{a}dt.$$

Calculating the integral we get:

$$\int_0^a \frac{te^{-st}}{a}dt = -\left.\frac{te^{-st}}{as}\right|_0^a + \frac{1}{as}\int_0^a e^{-st}dt$$

$$= -\frac{ae^{-as}}{as} + \frac{1}{as}\left.\left(-\frac{e^{-st}}{s}\right)\right|_0^a = -\frac{e^{-as}}{s} - \frac{e^{-as}}{as^2} + \frac{1}{as^2}$$

$$= \frac{1}{as^2} - \frac{(1+as)e^{-as}}{as^2}.$$

So,

$$\mathcal{L}(f(t)) = \left(\frac{1}{1-e^{-as}}\right)\left(\frac{1}{as^2} - \frac{(1+as)e^{-as}}{as^2}\right)$$

$$= \left(\frac{1}{1-e^{-as}}\right)\left(\frac{1-e^{-as}}{as^2}\right) - \frac{ase^{-as}}{as^2(1-e^{-as})}$$

$$= \frac{1}{as^2} - \frac{ae^{-as}}{as(1-e^{-as})} = \frac{1}{as^2} - \frac{e^{-as}}{s(1-e^{-as})}.$$

Inpulses and Delta Functions

7.6.1 - Solve the initial value problem

$$x'' + 4x = \delta(t);$$

$$x(0) = x'(0) = 0,$$

and graph the solution $x(t)$.

Solution - Taking the Laplace transform of both sides we get:

$$s^2X(s) + 4X(s) = 1$$

$$\Rightarrow X(s) = \frac{1}{s^2+4} = \frac{1}{2}\left(\frac{2}{s^2+4}\right).$$

So,

$$x(t) = \frac{1}{2}\sin(2t).$$

7.6.6 - Solve the initial value problem

$$x'' + 9x = \delta(t - 3\pi) + \cos 3t;$$
$$x(0) = x'(0) = 0,$$

and graph the solution $x(t)$.

Solution - Taking the Laplace transform of both sides:

$$s^2X(s) + 9X(s) = e^{-3\pi s} + \frac{s}{s^2+9}$$
$$\Rightarrow X(s) = \frac{e^{-3\pi s}}{s^2+9} + \frac{s}{(s^2+9)^2}.$$

The inverse Laplace transform is

$$\mathcal{L}^{-1}\left(\frac{e^{-3\pi s}}{s^2+9}\right) = \frac{1}{3}u(t-3\pi)\sin(3(t-3\pi)),$$
$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+9)^2}\right) = \frac{t\sin(3t)}{6}.$$

So,

$$x(t) = \frac{t\sin(3t) - 2u(t-3\pi)\sin(3(t-3\pi))}{6}.$$

7.6.11 - Apply Duhamel's principle to write an integral formula for the solution of the initial value problem

$$x'' + 6x' + 8x = f(t);$$

$$x(0) = x'(0) = 0.$$

Solution - Taking the Laplace transform of both sides we get:

$$s^2X(s) + 6sX(s) + 8X(s) = F(s),$$

where $\mathcal{L}(f(t)) = F(s)$.

So,

$$X(s) = \frac{F(s)}{s^2 + 6x + 8} = W(s)F(s),$$

where

$$W(s) = \frac{1}{s^2 + 6s + 8} = \frac{1}{(s+3)^2 - 1}.$$

We have

$$\mathcal{L}^{-1}(W(s)) = e^{-3t}\sinh(t),$$

and so,

$$x(t) = \mathcal{L}^{-1}(W(s)) * f(t) = \int_0^t e^{-3\tau}\sinh(\tau)f(t-\tau)d\tau.$$

7.6.14 - Verify that $u'(t-a) = \delta(t-a)$ by solving the problem

$$x' = \delta(t-a);$$
$$x(0) = 0$$

to obtain $x(t) = u(t-a)$.

Solution - Taking the Laplace transform of both sides we get

$$sX(s) = e^{-as}.$$

So,

$$X(s) = \frac{e^{-as}}{s}.$$

Calculating the inverse Laplace transform gives us:

$$\mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(e^{-as}\left(\frac{1}{s}\right)\right) = u(t-a)f(t-a),$$

where

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1.$$

So,

$$x(t) = u(t-a).$$

7.6.15 - This problem deals with a mass m on a spring (with constant k) that receives an impulse $p_0 = mv_0$ at time $t = 0$. Show that the initial value problems

$$mx'' + kx = 0;$$

$$x(0) = 0, x'(0) = v_0$$

and

$$mx'' + kx = p_0\delta(t);$$

$$x(0) = 0, x'(0) = 0$$

have the same solution. Thus the effect of $p_0\delta(0)$ is, indeed, to impart to the particle an initial momentum p_0.

Solution - The first problem has the solution

$$c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right).$$

We use the initial conditions to solve for the unknown coefficients c_1 and c_2. Plugging in $t = 0$ we get:

$$x(0) = c_1 = 0.$$

Calculating $x'(t)$ and plugging in $t = 0$ we get:

$$x'(t) = c_2\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right),$$

$$\Rightarrow x'(0) = c_2\sqrt{\frac{k}{m}} = v_0 \Rightarrow c_2 = v_0\sqrt{\frac{m}{k}}.$$

So,

$$x(t) = v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right).$$

On the other hand, if we examine the other differential equation involving the delta function, and take the Laplace transform of both sides we get:

$$ms^2X(s) + kX(s) = p_0$$

$$\Rightarrow X(s) = \frac{p_0}{ms^2 + k} = \frac{\frac{p_0}{m}}{s^2 + \frac{k}{m}}$$

$$= \frac{p_0}{m}\sqrt{\frac{m}{k}}\left(\frac{\sqrt{\frac{k}{m}}}{s^2 + \frac{k}{m}}\right) = v_0\sqrt{\frac{m}{k}}\left(\frac{\sqrt{\frac{k}{m}}}{s^2 + \frac{k}{m}}\right).$$

From this we get:

$$\mathcal{L}^{-1}(X(s)) = v_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right).$$

So, the ODEs have the same solution.