

# Math 2280 - Lecture 9

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In the last two lectures we've talked about differential equations for modeling populations. Today we'll return to the theme we touched upon in our second lecture, acceleration-velocity models, and see how differential equations can be used to model air resistance. We'll also discuss concepts with cool names, like "terminal velocity" and "escape velocity".

The problems for this section are:

Section 2.3 - 1, 2, 4, 10, 24

## Acceleration-Velocity Models

If an object is close to the earth's surface and we neglect any effects of air resistance, then the object will experience a constant downward force from gravity, and Newton's second law tells us that:

$$m \frac{dv}{dt} = F_G$$

where  $m$  is the object's mass,  $v$  is the object's velocity, and  $F_G$  is the constant force from gravity, which will be  $-mg$ .

This is very simple, and can be an OK model for some correspondingly simple physical situations, but in almost every real life situation we'll need to take air resistance into account. The phenomenon of air resistance is a pretty complicated one, which we won't discuss in depth, but we will look at some air resistance models that will give us a better idea of what goes on when an object falls on the earth's surface, and will also demonstrate mathematically some important behaviors we observe in real life.

## Resistance Proportional to Velocity

The first model we'll consider is the situation where air resistance is proportional to velocity, and in the direction opposite the direction of the velocity:

$$F_R = -kv$$

where  $k$  is a positive constant, and  $v$  is the object's velocity. Now, combining this air resistance with the (still assumed to be constant) force from gravity and using Newton's second law we get:

$$\begin{aligned} m \frac{dv}{dt} &= -kv - mg \\ \Rightarrow \frac{dv}{dt} &= -\rho v - g \end{aligned}$$

where  $\rho = \frac{k}{m}$ .

Example - Solve this differential equation.

$$\begin{aligned} \frac{dv}{dt} &= -\rho v - g \Rightarrow \frac{dv}{\rho v + g} = -dt \\ \Rightarrow \int \frac{dv}{\rho v + g} &= -\int dt \Rightarrow \frac{1}{\rho} \ln(\rho v + g) = -t + C \\ \Rightarrow \ln(\rho v + g) &= -\rho t + C \Rightarrow v(t) = \left( e^{-\rho t} - \frac{g}{\rho} \right) \\ v(0) = v_0 &= C - \frac{g}{\rho} \Rightarrow C = v_0 + \frac{g}{\rho} \\ \Rightarrow v(t) &= \left( v_0 + \frac{g}{\rho} \right) e^{-\rho t} - \frac{g}{\rho} = \boxed{\left( v_0 + \frac{g}{\rho} \right) e^{-\rho t} - \frac{g}{\rho}} \end{aligned}$$

We note that as  $t \rightarrow \infty$  our velocity approaches the value  $-\frac{g}{\rho}$ . This is called the object's *terminal velocity*. The absolute value of this is called the object's *terminal speed* and is given by:

$$|v_T| = \frac{mg}{k}$$

This phenomenon of terminal speed is what makes skydiving possible.

*Example* - A woman bails out of an airplane at an altitude of 10,000 ft, falls freely for 20s, then opens her parachute. How long will it take her to reach the ground? Assume linear air resistance  $\rho v$  ft/s<sup>2</sup>, taking  $\rho = .15$  without the parachute and  $\rho = 1.5$  with the parachute.

$$v(t) = \left(v_0 + \frac{g}{\rho}\right) e^{-\rho t} - \frac{g}{\rho}$$

$$\Rightarrow x(t) = -\frac{g}{\rho} t - \frac{1}{\rho} \left(v_0 + \frac{g}{\rho}\right) e^{-\rho t} + C$$

$$x(0) = x_0 \Rightarrow C = x_0 + \frac{1}{\rho} \left(v_0 + \frac{g}{\rho}\right) = x_0 + \frac{1}{\rho} (v_0 - v_T)$$

$$\Rightarrow x(t) = x_0 + v_T t + \frac{1}{\rho} (v_0 - v_T) (1 - e^{-\rho t})$$

$$x_0 = 10,000 \text{ ft} \quad v_0 = 0 \text{ ft/s} \quad v_T = -\frac{32.2 \text{ ft/s}^2}{-.15/\text{s}} \quad \rho = -.15/\text{s}$$

$$x(20) = 10,000 - \left(\frac{32.2}{-.15}\right)(20) + \frac{1}{.15} \left(0 + \frac{32.2}{-.15}\right) (1 - e^{-.15(20)})$$

$$\approx 7,067 \text{ ft}$$

$$v(20) = \left(0 + \frac{32.2}{-.15}\right) e^{-.15(20)} - \frac{32.2}{-.15} \approx -204 \text{ ft/s}$$

$$\text{Now, solve } 0 = 7,067 - \left(\frac{32.2}{1.5}\right) t_f + \left(\frac{1}{1.5}\right) (-204 + \frac{32.2}{1.5}) (1 - e^{-1.5 t_f})$$

$$\Rightarrow t_f \approx 340 \text{ s.} \quad \text{So, about } 340 \text{ s} + 20 \text{ s} \approx \boxed{6 \text{ min}}$$

## Variable Gravitational Acceleration

The model of constant gravitation only works when we're close to the surface of the earth, and the distances we're dealing with are small relative to the radius of the earth. If we start to deal with larger distances, then we must take into account that acceleration from gravity is weaker the farther we are away from the earth. Newton's law of universal gravitation tells us that the force from gravity experienced a distance  $r$  from the center of the earth will be:

$$F = -\frac{GmM}{r^2}$$

where  $m$  is the mass of the object,  $M$  is the mass of the earth,  $G$  is Newton's gravitational constant  $G = 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2$ .

We can use this relation to calculate an object's escape velocity on the surface of the earth. This is the speed at which an object must be moving away from the earth at the earth's surface if it is to break free from the gravitational attraction of the earth and continue to move away "forever".

Well, we note that if we move away from the earth along a line that goes through the earth's center, then Newton's second law tells us:

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}.$$

If we note that  $v = \frac{dr}{dt}$  then we can transform this relation into:

$$v \frac{dv}{dr} = -\frac{GM}{r^2}.$$

If we integrate both sides with respect to  $r$  we get:

$$\frac{1}{2}v^2 = \frac{GM}{r} + C.$$

If we say  $v(0) = v_0$  and solve this for  $C$  we get:

$$v^2 = v_0^2 + 2GM \left( \frac{1}{r} - \frac{1}{R} \right).$$

If the object is to escape from the "clutch" of the earth then its velocity must always be positive as  $r \rightarrow \infty$ . This is possible if

$$v_0 \geq \sqrt{\frac{2GM}{R}}$$

So, the escape velocity for the earth (or for any planet of given mass  $M$ ) is:

$$v_0 = \sqrt{\frac{2GM}{R}}$$

For the earth the escape velocity is  $v_0 \approx 11,180\text{m/s}$ .

*Example* - Suppose that you are stranded - your rocket engine has failed - on an asteroid of diameter 3 miles, with density equal to that of the earth with radius 3960 miles. If you have enough spring in your legs to jump 4 feet straight up on earth while wearing your space suit, can you blast off from this asteroid using leg power alone?

$$v_E^2 = \frac{2GM_E}{R_E} \Rightarrow \frac{3v_E^2}{8\pi GR_E^2} = \frac{M_E}{\frac{4}{3}\pi R_E^3}$$

$$\Rightarrow \frac{3v_E^2}{8\pi GR_E^2} = \frac{3v_A^2}{8\pi GR_A^2} \Rightarrow \left(\frac{R_A}{R_E}\right)^2 = \left(\frac{v_A}{v_E}\right)^2$$

$$\Rightarrow v_A = v_E \left(\frac{R_A}{R_E}\right) = 11,180\text{m/s} \left(\frac{1.5\text{ miles}}{3,960\text{ miles}}\right) = 4.24\text{m/s}$$

$$v_{\text{Jump}} = \sqrt{2(9.8\text{m/s}^2)(4\text{ft})\left(\frac{1\text{m}}{3.28\text{ft}}\right)} = 4.89\text{m/s}$$

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So, yes, you can get off!