

Math 2280 - Lecture 7: Exact Equations and Population Models

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Today we're going to finish our discussion of section 1.6 by discussing "exact" differential equations, and then we're going to explore one of the major applications of differential equations - population models.

The exercises for section 1.6 are:

Section 1.6 - 1, 3, 13, 16, 22, 26, 31, 36, 56

The exercises for section 2.1 are:

Section 2.1 - 1, 8, 11, 16, 29

Exact Differential Equations

We've seen in our solutions to differential equations that sometimes, frequently even, the solution is not an explicit equation describing y as a function of x , but is instead an implicit function of the form

$$F(x, y) = C,$$

where the dependence of y on x is implicit. We can recover our initial differential equation by differentiating both sides with respect to x , and then solving for dy/dx :

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0;$$

solving for dy/dx :

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

We can write the first equation above a bit more symmetrically as

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0.$$

Turning this around, suppose we're given a differential equation in the form

$$M(x, y)dx + N(x, y)dy = 0.$$

If the functions $M(x, y)$ and $N(x, y)$ are such that there exists a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y),$$

and

$$\frac{\partial F}{\partial y} = N(x, y),$$

then the (implicit) solution to the differential equation will be

$$F(x, y) = C.$$

Recall from multivariable calculus that if the mixed partial derivatives F_{xy} and F_{yx} are continuous in an open subset of the xy -plane, then they're equal on that subset. In practice F_{xy} and F_{yx} are usually continuous, and so if $F_x = M$ and $F_y = N$ then we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This is a *necessary* condition. Turns out it's also sufficient.

Example - Solve the differential equation

$$(2xy^2 + 3x^2)dx + (2x^2y + 4y^3)dy = 0.$$

$$\frac{d}{dy} (2xy^2 + 3x^2) = 4xy$$

$$\frac{d}{dx} (2x^2y + 4y^3) = 4xy$$

So, it's exact.

$$\int [2xy^2 + 3x^2] dx = x^2y^2 + x^3 + g(y)$$

$$\frac{d}{dy} (x^2y^2 + x^3 + g(y)) = 2x^2y + g'(y) = 2x^2y + 4y^3$$

$$\Rightarrow g'(y) = 4y^3 \Rightarrow g(y) = y^4 + C.$$

So,

$$F(x, y) = \cancel{2x^2} x^2y^2 + x^3 + y^4 + C$$

The solution to the ODE is

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$$\boxed{x^2y^2 + x^3 + y^4 = C}$$

Population Models

Population growth models depends upon two parameters - birth and death. We call β the “birth rate” of the population, and δ the “death rate” of the population, and the population model is

$$\frac{dP}{dt} = (\beta - \delta)P.$$

The terms β and δ are not necessarily constants, and could themselves be functions of time or the population size.

The simplest population model is one in which β and δ are constant. In this scenario the rate of growth of the population is directly proportional to the population. A population with this characteristic is modeled by the differential equation

$$\frac{dP}{dt} = kP.$$

There are many things in life that actually do fit this model¹, and it can be a reasonable short-term model for population growth. However, the thing about exponentials is they grow really, really fast, and no real population can keep growing exponentially forever. Something, usually the exhaustion of resources, enters the picture to correct things.

So, let’s look at a slightly more sophisticated population model. Suppose the birth rate of our population is a *linear* decreasing function of the population size. So,

$$\beta = \beta_0 - \beta_1 P,$$

while the death rate is constant,

$$\delta = \delta_0.$$

¹Compound interest, for example.

In this case the population model will be

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0)P,$$

which we can rewrite as

$$\frac{dP}{dt} = aP - bP^2.$$

If the parameters a and b are both positive then the above equation is called the *logistic equation*. It's actually even more convenient to rewrite it as

$$\frac{dP}{dt} = kP(M - P)$$

where $k = b$ and $M = a/b$.

Example - Solve the logistic growth equation with initial condition $P(0) = P_0 > 0$.

$$\frac{dP}{dt} = kP(M - P)$$

$$\Rightarrow \frac{dP}{P(M - P)} = k dt \Rightarrow \left(\frac{\left(\frac{1}{M}\right)}{P} + \frac{\left(-\frac{1}{M}\right)}{(M - P)} \right) dP = \frac{k dt}{\cancel{M}}$$

Partial fraction decomposition

$$\Rightarrow \left(\frac{dP}{P} - \frac{dP}{M - P} \right) = \int M k dt$$

$$\ln(P) - \ln(M - P) = Mkt + C \Rightarrow \ln\left(\frac{P}{M - P}\right) = Mkt + C$$

More space for the example problem.

$$\Rightarrow \ln\left(\frac{M-p}{p}\right) = -Mkt + C$$

$$\Rightarrow \frac{M-p}{p} = C e^{-Mkt}$$

$$\Rightarrow \frac{M}{p} - 1 = C e^{-Mkt}$$

$$\Rightarrow \frac{M}{p} = 1 + C e^{-Mkt}$$

$$\Rightarrow P(t) = \frac{M}{1 + C e^{-Mkt}}$$

$$P(0) = P_0 = \frac{M}{1+C} \Rightarrow P_0 + P_0 C = M \Rightarrow C = \frac{M}{P_0} - 1$$

$$P(t) = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right) e^{-Mkt}} = \frac{M P_0}{P_0 + (M - P_0) e^{-Mkt}}$$

$$P(t) = \frac{M P_0}{P_0 + (M - P_0) e^{-Mkt}}$$

Consider a population $P(t)$ of unsophisticated animals in which females rely solely on chance encounters to meet males for reproductive purposes.² Here the rate of growth will be proportional to the product of the number of males, $P/2$, and the number of females, $P/2$. So, births will occur at a rate kP^2 , and the "birth rate" will be kP . If the death rate, δ , is considered to be constant then the differential equation modeling our population will be

$$\frac{dP}{dt} = kP^2 - \delta P = kP(P - M).$$

This type of differential equation exhibits very interesting behavior.

Example - Solve the differential equation above, and explain why it's sometimes called the "Doomsday-Extinction" model.

$$\frac{dP}{dt} = kP(P-M)$$

$$\frac{dP}{P(P-M)} = k dt \Rightarrow \left(\frac{\frac{1}{M}}{P-M} - \frac{\frac{1}{M}}{P} \right) dP = k dt$$

$$\Rightarrow \int \frac{dP}{P-M} - \frac{dP}{P} = \int M k dt$$

$$\ln(P-M) - \ln(P) = M k t + C$$

$$\Rightarrow \ln\left(\frac{P-M}{P}\right) = M k t + C$$

²For example, bars on weekends...

More space for the example problem.

$$\Rightarrow \frac{P-M}{P} = C e^{Mkt}$$

$$\Rightarrow 1 - \frac{M}{P} = C e^{Mkt} \Rightarrow 1 - C e^{Mkt} = \frac{M}{P}$$

$$P(t) = \frac{M}{1 - C e^{Mkt}} \quad P(0) = P_0$$

$$P_0 = \frac{M}{1-C} \Rightarrow 1-C = \frac{M}{P_0} \Rightarrow C = 1 - \frac{M}{P_0}$$

$$P(t) = \frac{M}{1 - \left(1 - \frac{M}{P_0}\right) e^{Mkt}} = \boxed{\frac{M P_0}{P_0 - (P_0 - M) e^{Mkt}}}$$

If $P_0 < M$ then $\lim_{t \rightarrow \infty} P(t) = 0$

If $P_0 > M$ then $\lim_{t \rightarrow t_d^-} P(t) = \infty$

$$P_0 = (P_0 - M) e^{Mkt} \quad t_d = \frac{\ln\left(\frac{P_0}{P_0 - M}\right)}{Mk}$$

t_d is "Doomsday", the finite amount of time before we have a vertical asymptote.