Math 2280 - Lecture 6: Substitution Methods for First-Order ODEs

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Spring 2013

In today's lecture we're going to examine another technique that can be useful for solving first-order ODEs. Namely, substitutuion. Now, as with u-substitutuion in calculus, figuring out the right substitution to make, assuming there even is a right substitutuion, is not always obvious. It's part science, and part art. However, there are a few general forms that we can recognize, and for which there are straightforward methods. We'll discuss some of these today.

This will be the first of two lectures on section 1.6 from the textbook. In the next lecture will also begin discussing section 2.1, and then in the lecture after that will finish 2.1 and cover 2.2.

The exercises for this section are:

Section 1.6 - 1, 3, 13, 16, 22, 26, 31, 36, 56

The Idea of Substitution

Suppose we're given the first-order differential equation in standard form that we've all learned to know and love:

$$\frac{dy}{dx} = f(x, y).$$

The function $f(x, y)$ may contain a combination of the variables x and y:

$$v = \alpha(x, y)$$

that suggests itself as a new independent variable v.

If we can solve $v = \alpha(x, y)$ for y in terms of x and v:

$$y = \beta(x, y),$$

then the chain rule gives us

$$\frac{dy}{dx} = \beta_x + \beta_v \frac{dv}{dx}.$$

Using this, and the relation

$$\frac{dy}{dx} = f(x, y)$$

we can get a *new* differential equation

$$\frac{dv}{dx} = g(x, v)$$

which we might be able to solve for v, and from there we can solve for y. No problem, right?[1]

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[1] Yeah, it probably looks a little scary now. It will (hopefully) become more clear and less frightening once you've worked through some examples.

Example - Find a general solution to the differential equation

$$yy' + x = \sqrt{x^2 + y^2}.$$

$v = x^2 + y^2$

$v' = 2x + 2yy'$

$\Rightarrow \quad v' = 2\frac{y}{h}\sqrt{v}$

$\Rightarrow \int \frac{dv}{\sqrt{v}} = \int \frac{2dx}{x}$

$\Rightarrow 2\sqrt{v} = \frac{2x}{x} + C$

$\Rightarrow \sqrt{v} = \frac{x}{x} + C$

$v = \left(\frac{x}{x} + C\right)^2$

$\Rightarrow \boxed{x^2 + y^2 = (x + C)^2}$

$\Rightarrow x^2 + y^2 = x^2 + 2xC + C^2$

$\Rightarrow y^2 = 2xC + C^2 \qquad or \qquad \boxed{x - \sqrt{x^2 + y^2} = C}$

Homogeneous Equations

A *homogeneous* first-order differential equation is one that can be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

If we make the substitutuion $v = y/x$ then we can transform our equation into a separable equation

$$x\frac{dv}{dx} = F(v) - v.$$

Example - Find the general solution to the differential equation

$$(x+y)y' = x - y.$$

$$y' = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}} \implies x\frac{dv}{dx} = \frac{1-v}{1+v} - v$$

$$\implies x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v} \implies \int\left(\frac{1+v}{1-2v-v^2}\right)dv = \int\frac{dx}{x}$$

$$\implies \begin{array}{l} u = 1-2v-v^2 \\ du = (-2-2v)dv \end{array} \qquad \frac{1}{u} = Cx^2$$

$$-\frac{1}{2}\int\frac{du}{u} = \int\frac{dx}{x} \qquad 1 = Cx^2\left(1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right)$$

$$\implies \ln\left(\frac{1}{\sqrt{u}}\right) = \ln(x) + C \qquad \implies \boxed{C = x^2 - 2xy - y^2}$$

$$\implies \frac{1}{\sqrt{u}} = Cx$$

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Note: I'm, as usual, playing fast and loose with the unknown C here

Bernoulli Equations

A Bernoulli equation[2] is a first-order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

If $n = 0$ or $n = 1$ then it's just a linear differential equation. Otherwise, if we make the substitution

$$v = y^{1-n}$$

the differential equation above transforms into the linear equation

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x),$$

which we can then solve.

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[2]Named after one member of that famous 17th-century mathematical family. I'm not sure which.

Example - Find the general solution to the differential equation

$$xy' + 6y = 3xy^{4/3}.$$

$$\Rightarrow \frac{dy}{dx} + \frac{6}{x} y = 3 y^{4/3}$$

$$v = y^{1-n}$$

$$\frac{dv}{dx} + \left(-\frac{1}{3}\right)\left(\frac{6}{x}\right) v = \left(-\frac{1}{3}\right)(3)$$

$$\Rightarrow \frac{dv}{dx} - \frac{2}{x} v = -1 \qquad e^{-\int \frac{2}{x} dx} = \frac{1}{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{v}{x^2}\right) = -\frac{1}{x^2}$$

$$\Rightarrow \frac{v}{x^2} = \frac{1}{x} + C \qquad \Rightarrow \cancel{v = (x^2 - x)^3}$$

$$\qquad \qquad V(x) = (x^2 + x)$$

$$\Rightarrow y^{-1/3} = \cancel{(x^2 - x)^3}\ (x^2 + x)$$

$$\boxed{\cancel{y(x) = \frac{1}{((x^2 - x)^3)^3}}} \qquad \boxed{y(x) = \frac{1}{((x^2 + x))^3}}$$

Example - Suppose an airplane departs from the point $(a, 0)$ located due east of its intended destination - an airport located at the origin $(0, 0)$. The plane travels with constant speed v_0 relative to the wind, which is blowing due north with constant speed w. Assume the plane's pilot maintains its heading directly towards the origin. What is the flight's trajectory?

$$\frac{dx}{dt} = -\frac{v_0 x}{\sqrt{x^2 + y^2}} \qquad \frac{dy}{dt} = w - \frac{v_0 y}{\sqrt{x^2 + y^2}}$$

$$\frac{dy}{dx} = \frac{1}{v_0 x}\left(v_0 y - w\sqrt{x^2 + y^2} \right)$$

$$k = \frac{w}{v_0} \implies \frac{dy}{dx} = \frac{y}{x} - k\left[1 + \left(\frac{y}{x}\right)^2 \right]^{1/2}$$

$$y = xv \qquad y' = v + xv'$$

$$\implies v + xv' = v - k\left[1 + v^2 \right]^{1/2}$$

$$\implies v' = -\frac{k}{x}(1 + v^2)^{1/2} \implies \int \frac{dv}{\sqrt{1+v^2}} = -\int \frac{k}{x} dx$$

$$\implies \ln\left(v + \sqrt{1+v^2} \right) = -k \ln x + C$$

$$v(a) = \frac{y(a)}{a} = 0 \implies C = k \ln(a)$$

$$\implies v = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k} \right]$$

$$\boxed{y(x) = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k} \right]}$$