Math 2280 - Lecture 4: Separable Equations and Applications

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For the last two lectures we've studied first-order differential equations in standard form

$$y' = f(x, y).$$

We learned how to solve these differential equations for the special situation where $f(x, y)$ is independent of the variable y, and is just a function of x, $f(x)$. We also learned about slope fields, which give us a geometric method for understanding solutions and approximating them, even if we cannot find them directly.

Today we're going to discuss how to solve first-order differential equations in standard form in the special situation where the function $f(x, y)$ is *separable*, which means we can write $f(x, y)$ as the product of a funciton of x, and a function of y.

The exercises for this section are:

Section 1.4 - 1, 3, 17, 19, 31, 35, 53, 68

Separable Equations and How to Solve Them

Suppose we have a first-order differential equation in standard form:

$$\frac{dy}{dx} = h(x, y).$$

If the function $h(x, y)$ is *separable* we can write it as the product of two functions, one a function of x, and the other a function of y. So,

$$h(x, y) = \frac{g(x)}{f(y)}.$$

In this situation we can manipulate our differtial equation to put everything with a y term on one side, and everything with an x term on the other:

$$f(y)dy = f(x)dx.$$

From here we can just integrate both sides of the equation, and then solve for y as a funciton of x!

So, for example, suppose we're given the differential equation

$$\frac{dP}{dt} = P^2.$$

We can rewrite this equation as

$$\frac{dP}{P^2} = dt,$$

and then integrate both sides of the equation to get

$$-\frac{1}{P} = t + C.$$

Solving this for P as a function of t gives us

$$P(t) = \frac{1}{C - t}.$$[1]

Note that this function has a vertical asymptote as t approaches C. If this is a population model, this is called *doomsday*!

Examples of Separable Differential Equations

Suppose we're given the differential equation

$$\frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5}.$$

This differential equation is separable, and we can rewrite it as

$$(3y^2 - 5)dy = (4 - 2x)dx.$$

If we integrate both sides of this differential equation

$$\int (3y^2 - 5)dy = \int (4 - 2x)dx$$

we get

$$y^3 - 5y = 4x - x^2 + C.$$

This *is* a solution to our differential equation, but we cannot readily solve this equation for y in terms of x. So, our solution to this differential equation must be implicit.

[1]Note that we're playing a little fast and loose with the unknown constant C here. In particular, if we multiply an unknown constant C by -1, it's still just an unknown constant, and we continue to call it (positive) C.

If we're given an initial value, say $y(1) = 3$, then we can easily solve for the unknown constant C:

$$3^3 - 5(3) = 4(1) - 1^2 + C \Rightarrow C = 9.$$

So, around the point $(1, 3)$ the differential equation will have the unique solution given implicitly by the curve defined by

$$y^3 - 5y = 4x - x^2 + 9.$$

Example - Find all solutions to the differential equation

$$\frac{dy}{dx} = 6x(y-1)^{\frac{2}{3}}.$$

$$\int \frac{dy}{(y-1)^{2/3}} = \int 6x\,dx$$

$$3(y-1)^{1/3} = 3x^2 + C$$

$$\Rightarrow (y-1)^{1/3} = x^2 + C$$

$$\Rightarrow \boxed{y = (x^2 + C)^3 + 1}$$

Now, the function $y(x) = 1$ is also a solution! If we're given the initial value problem $y(0) = 1$ then we have 2 solutions: $y_1(x) = x^6 + 1$ and $y_2(x) = 1$.

So, what's going on here? Turn the page to find out.

More room for the example.

The function

$$f(x,y) = 6x\,(y-1)^{2/3}$$

is continuous everywhere, but

$$\frac{\partial f}{\partial y} = \frac{4x}{(y-1)^{1/3}}$$

is undefined where $y=1$. So, for any initial value $y(a)=b$ if $b \neq 1$ there is a unique local solution, but if $b=1$ there is not.

A very common, and simple, type of differential equation that is used to model many, many things[2] is

$$\frac{dx}{dt} = kx$$

where k is some constant.

Now, this is a separable equation, and so it can be solved by our methods. First, we rewrite it as

$$\frac{dx}{x} = kdt,$$

and then integrate both sides

$$\int \frac{dx}{x} = \int kdt$$

to get

$$\ln x = kt + C.$$

If we then exponentiate both sides we get

$$x(t) = e^{kt+C} = e^C e^{kt} = Ce^{kt}.^3$$

So, the solution to our differential equation is exponential growth (if $k > 0$) or exponential decay (if $k < 0$). If $k = 0$ the answer is just a boring unknown constant.

[2]Compound interest, population growth, radioactive decay, etc...

[3]The American Society for the Prevention of Notation Abuse would strongly protest this last equality. I'm just saying that e^C, where C is an unknown constant, is itself just an unknown constant, and I don't like having to come up with new letters, so I just continue to represent the unknown constant as C.

Radioactive decay is quite accurately measured by an exponential decay function. For ^{14}C decay, the decay constant is $k \approx -0.0001216$ if t is measured in years.

Example - Carbon taken from a purported relic of the time of Christ contained 4.6×10^{10} atoms of ^{14}C per gram. Carbon extracted from a present-day specimen of the same substance contained 5.0×10^{10} atoms of ^{14}C per gram. Compute the approximate age of the relic. What is your opinion as to its authenticity?

$$C(t) = C_0 e^{kt}$$

$$C(t_0) = 4.6 \times 10^{10}$$

$$C_0 = 5.0 \times 10^{0}$$

$$k = -0.0001216$$

$$\Rightarrow \quad t_0 = \frac{\ln\left(\frac{4.6 \times 10^{10}}{5.0 \times 10^{10}}\right)}{-0.0001216} \approx 685.7 \text{ years.}$$

So, probably not from the time of Christ (about 2,000 years ago.)