Math 2280 - Lecture 43

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Today I'm not going to introduce any new material. Instead, I'm just going to work some difficult problems because I know many of you are struggling with some of the material from chapters 8 and 9. Chapter 8 in particular. So, I figured it might be helpful to see a few more problems worked out in detail.

Series Solutions Near Ordinary Points

Example - Find the general solution to the differential equation below in terms of power series in x.

$$y'' - x^2 y' - 3xy = 0.$$

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Solution - We first note that as the coefficient in front of y'' is a constant, and the coefficients in front of the lower order terms are polynomials, the functions $P(x) = -x^2$ and $Q(x) = -3x$ are analytic at $x = 0$, and so 0 is an ordinary point of the differential equation.

So, we assume that we can find a solution of the form:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

If we plug these values into our differential equation we get:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x^2 \sum_{n=1}^{\infty} nc_n x^{n-1} - 3x \sum_{n=0}^{\infty} c_n x^n = 0.$$

Multiplying through this becomes:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^{n+1} - \sum_{n=0}^{\infty} 3c_n x^{n+1} = 0$$

Now, the only place we're going to have an x^0 term is in the first sum, when $n = 2$, and so given the x^0 coefficient must be 0 we get $c_2 = 0$.

For higher order terms all of our sums are going to enter into the calculation, and if we shift the index on the first one by 3, and note that starting the second series at 0 is the same as starting it at 1, we get:

$$\sum_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} - nc_n - 3c_n]x^{n+1} = 0.$$

The coefficient of x^{n+1} must be 0 for all n, and so solving this recursion relation gives us:

$$c_{n+3} = \frac{c_n}{n+2}.$$

We note this relation places no restriction on ther terms c_0 and c_1, so these terms are arbitrary, and represent the arbitrary constants in our general solution.[1]

[1]"Arbitrary" is perhaps the wrong word here. More precisely, they'd be determined by the initial conditions of our system.

Now, in some sense we're done, in that we've figured out how to represent every term in our power series in terms of our two arbitrary constants c_0 and c_1. However, if possible we'd like to derive a closed form solution for our terms.

If we take a look at our first few terms we get:

$$
\begin{aligned}
c_3 &= \frac{c_0}{2}, \\
c_4 &= \frac{c_1}{3}, \\
c_5 &= \frac{c_2}{4} = 0, \\
c_6 &= \frac{c_3}{5} = \frac{c_0}{5 \times 2}, \\
c_7 &= \frac{c_4}{6} = \frac{c_1}{6 \times 3}, \\
c_8 &= \frac{c_5}{7} = 0, \\
c_9 &= \frac{c_6}{8} = \frac{c_0}{8 \times 5 \times 2}, \\
c_{10} &= \frac{c_7}{9} = \frac{c_1}{9 \times 6 \times 3}, \\
&\vdots
\end{aligned}
$$

From these first few terms we can already see our pattern emerging. This pattern is:

$$
\begin{aligned}
c_{3n} &= \frac{c_0}{(3n-1) \times (3n-4) \times \cdots \times 5 \times 2} \qquad \text{for } n > 0, \\
c_{3n+1} &= \frac{c_1}{3n \times (3n-3) \times \cdots \times 6 \times 3} = \frac{c_1}{3^n n!}, \\
c_{3n+2} &= 0.
\end{aligned}
$$

So, our closed form solution is:

$$y(x) = c_0\left(1 + \sum_{n=1}^{\infty}\frac{x^{3n}}{(3n-1)\times(3n-4)\times\cdots\times 5\times 2}\right) + c_1\sum_{n=0}^{\infty}\frac{x^{3n+1}}{3^n n!}.$$

Series Solutions Near Regular Singular Points

Example - Find the general solution to the differential equation:

$$2xy'' - y' - y = 0.$$

Solution - First we note that if we divide everything through by $2x$ the coefficients in front of both y' and y are not analytic at $x = 0$. However, $p(x) = xP(x) = -1/2$ and $q(x) = x^2Q(x) = -x/2$ are, so $x = 0$ is a regular singular point.

The constant term of $p(x)$ is $-1/2$, while the constant term of $q(x)$ is 0. So, the indicial equation is:

$$r(r-1) - \frac{1}{2}r = 0,$$

which has solutions $r = 0$ and $r = 3/2$. As the difference between these two solutions is not an integer, we know we'll be able to find two Frobenius series solutions.

Our Frobenius series solutions will be of the form:

$$y(x) = x^r\sum_{0}^{\infty}c_nx^n = \sum_{n=0}^{\infty}c_nx^{n+r},$$

$$y'(x) = \sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1},$$

$$y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}.$$

Plugging these into our differential equation gives us:

$$\sum_{0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - \sum_{0}^{\infty} c_n x^{n+r} = 0.$$

Now, the lowest order term is x^{r-1}, and the coefficient there is:

$$[r(r-1) - r/2]c_0.$$

For our two values of r this is automatically satisfied for any c_0 (it's just c_0 multiplied by the indicial equation, which our two values of r solve by definition), and so c_0 is arbitrary.

For the higher order terms, simplifying the above series gives us:

$$\sum_{n=1}^{\infty}[(2(n+r+1)(n+r) - (n+r+1))c_{n+1} - c_n]x^{n+r} = 0.$$

Each of the x^{n+r} coefficients must be 0, and so we get the recursion relation:

$$c_{n+1} = \frac{c_n}{2(n+r+1)(n+r) - (n+r+1)}.$$

Plugging in our values of r we get, for $r = 0$:

$$c_{n+1} = \frac{c_n}{2(n+1)n - (n+1)} = \frac{c_n}{(2n-1)(n+1)}.$$

We note that for no integer value of $n \geq 1$ is the denominator 0, so this defines our series for all values of n for which we're interested.

If we plug in $r = 3/2$ we get:

$$c_{n+1} = \frac{c_n}{(n+1)(2n+5)}.$$

We again note that for no integer value of $n \geq 1$ is the denominator 0, so this again defines our series for all n in which we're interested.

Now, as mentioned earlier, we could say we're done here. However, let's see if we can find a closed form solution for our series.

if we examine the $r = 0$ terms we get:

$$c_1 = -\frac{c_0}{1},$$
$$c_2 = -\frac{c_1}{2} = -\frac{c_0}{2},$$
$$c_3 = -\frac{c_2}{3 \times 3} = -\frac{c_0}{(3 \times 1) \times (3 \times 2 \times 1)},$$
$$c_4 = -\frac{c_3}{5 \times 4} = -\frac{c_0}{(5 \times 3 \times 1) \times (4 \times 3 \times 2 \times 1)},$$
$$\vdots$$

We can see the pattern here, and so we get:

$$c_n = -\frac{c_0}{((2(n-1)-1) \times (2(n-1)-3) \times \cdots \times 3 \times 1) \times (n \times (n-1) \times \cdots \times 2 \times 1)}$$
$$= -\frac{c_0 2^{n-1}(n-1)!}{(2(n-1))!n!} = -\frac{c_0 2^{n-1}}{n \times (2(n-1))!}.$$

So, one of our Frobenius solutions is:

$$y_1(x) = c_0 \left(1 - \sum_{n=1}^{\infty} \frac{x^n 2^{n-1}}{n \times (2(n-1))!} \right).$$

On the other hand, for our $r = 3/2$ series we get:

$$c_1 = \frac{c_0}{1 \times 5},$$

$$c_2 = \frac{c_1}{2 \times 7} = \frac{c_0}{(2 \times 1) \times (7 \times 5)} = \frac{3c_0}{(2 \times 1) \times (7 \times 5 \times 3 \times 1)},$$

$$c_3 = \frac{c_2}{3 \times 9} = \frac{3c_0}{(3 \times 2 \times 1) \times (9 \times 7 \times 5 \times 3 \times 1)},$$

$$\vdots$$

We can already see where this is going. The denominator in each term is just $n!$ multiplied by the product of the first $n+2$ odd terms. Writing this out in closed form we get:

$$c_n = \frac{3c_0}{\left(\frac{(2(n+2))!}{2^{n+2}(n+2)!}\right) n!} = \frac{3(n+2)(n+1)2^{n+2}c_0}{(2(n+2))!}.$$

So, our second Frobenius solution is:

$$y_2(x) = c_0 x^{\frac{3}{2}} \left(1 + \sum_{n=1}^{\infty} \frac{3(n+2)(n+1)2^{n+2}x^n}{(2(n+2))!} \right).$$

Using these we see, when the smoke clears, our general solution is:

$$y(x) = c_0 \left(1 - \sum_{n=1}^{\infty} \frac{x^n 2^{n-1}}{n \times (2(n-1))!} \right) + c_1 x^{\frac{3}{2}} \left(1 + \sum_{n=1}^{\infty} \frac{3(n+2)(n+1)2^{n+2}x^n}{(2(n+2))!} \right).$$

Where we've renamed our second constant c_1 so as to not give it the same symbol as our first constant.

Calculating Fourier Series

Example - Calculate the Fourier series for the periodic function $f(t)$ where one period is given by:

$$f(t) = t, \qquad -2 < t < 2.$$

Solution - This function is periodic with period 4, so $L = 4/2 = 2$. So, the Fourier series for this function will be:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{2}\right) + b_n \sin\left(\frac{n\pi t}{2}\right) \right),$$

where the constants are given by:

$$a_n = \frac{1}{2}\int_{-2}^{2} t\cos\left(\frac{n\pi t}{2}\right)dt;$$
$$b_n = \frac{1}{2}\int_{-2}^{2} t\sin\left(\frac{n\pi t}{2}\right)dt.$$

For any of the a_n terms we note that the integrand is an odd function, and so the integral will be 0. As for the b_n we get:

$$b_n = \frac{1}{2}\int_{-2}^{2} t\sin\left(\frac{n\pi t}{2}\right)dt = \int_{0}^{2} t\sin\left(\frac{n\pi t}{2}\right)dt.$$

Integrating these by parts we get:

$$\int t\sin\left(\frac{n\pi t}{2}\right)dt = \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi t}{2}\right) - \frac{2}{n\pi}t\cos\left(\frac{n\pi t}{2}\right).$$

If we evaluate this at the appropriate limits we get:

$$b_n = \int_0^2 t \sin\left(\frac{n\pi t}{2}\right) dt = -\frac{4}{n\pi}(-1)^n = \frac{4(-1)^{n+1}}{n\pi}.$$

So, our Fourier series will be:

$$\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi t}{2}\right)$$

$$= \frac{4}{\pi}\left(\sin\left(\frac{\pi t}{2}\right) - \frac{1}{2}\sin\left(\frac{2\pi t}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{2}\right) - \cdots\right).$$