Math 2280 - Lecture 42

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In today's lecture, the last one of the semester in which I'll be introducing new material, we're going to talk about the wave equation. The wave equation is a partial differential equation that models the motion of a plucked string. The textbook goes over a derivation of the equation, but we'll just cut to the chase and state that the *one-dimensional wave equation* is:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

Today, we'll learn how this equation is solved.

This lecture corresponds with section 9.6 from the textbook. The assigned problems from this section are:

Section 9.6 - 1, 3, 5, 7, 14

Vibrating Strings and the One-Dimensional Wave Equation

The one-dimensional wave equation models the movement of a plucked string. This movement is represented by a function $y(x,t)$, which represents the vertical displacement of the string at the horizontal distance x and at the time t. Frequently the ends of the string are tied down and not allowed to move, so we have the endpoint conditions

$$y(0,t) = y(L,t) = 0.$$

If we specify the initial position of the string, $y(x,0) = f(x)$, and the initial velocity $y_t(x,0) = g(x)$ then we have a boundary value problem that we'd like to solve. Bringing this all together, we want to solve the boundary value problem

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad & (0 < x < l, t > 0); \\
y(0,t) = y(L,t) = 0&, \\
y(x,0) = f(x) \qquad & (0 < x < L), \\
y_t(x,0) = g(x) \qquad & (0 < x < L).
\end{aligned}$$

Our method is the same method we used for solving the heat equation. Namely, we use separation of variables. So, we assume that our solution is of the form

$$y(x,t) = X(x)T(t).$$

If we plug this into our differential equation we get the relation

$$\frac{X''}{X} = \frac{T''}{a^2 T}.$$

This is only possible if both functions are equal to the same constant. So, we write

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

for some constant λ. So, our partial differential equation separates into the two ordinary differential equations

$$X'' + \lambda X = 0,$$

and

$$T'' + \lambda a^2 T = 0.$$

The endpoint conditions require $X(0) = X(L) = 0$ if $T(t)$ is non-trivial, and so $X(x)$ must satisfy the eigenvalue problem we're grown to know and love

$$X'' + \lambda X = 0,$$
$$X(0) = X(L) = 0.$$

We solved this two lectures ago. The eigenvalues of this problem are the numbers

$$\lambda_n = \frac{n^2\pi^2}{L^2},$$

where n is a positive integer. The associated eigenfunctions are

$$X_n(x) = \sin\frac{n\pi x}{L}.$$

Now, what about the functions $T_n(t)$? Well, we first note that we can solve our boundary value problem if we can find two solutions, one to the boundary value problem

$$y_{tt} = a^2 y_{xx};$$
$$y(0,t) = y(L,t) = 0,$$
$$y(x,0) = f(x),$$
$$y_t(x,0) = 0,$$

and the other to the boundary value problem

$$y_{tt} = a^2 y_{xx};$$
$$y(0,t) = y(L,t) = 0,$$
$$y(x,0) = 0,$$
$$y_t(x,0) = g(x).$$

If we call these two solutions y_A and y_B then our final solution will be $y = y_A + y_B$.

If we're solving for y_A then our equation for $T_n(t)$ will be

$$T_n'' + \frac{n^2\pi^2}{L^2} T_n = 0,$$

with initial condition $T_n'(0) = 0$. The general solution to this differential equation will be a multiple of

$$T_n(t) = \cos \frac{n\pi a t}{L}.$$

Our final solution for y_A will then be

$$y_A(x,t) = \sum_{n=1}^{\infty} A_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}.$$

We need the coefficients A_n to satisfy

$$y_A(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = f(x),$$

and so

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

As for the solution y_B, the ODE for $T_n(t)$ here will be

$$T_n'' + \frac{a^2n^2\pi^2}{L^2}T_n = 0,$$

with initial condition $T_n(0) = 0$. Runnning through our standard arguments we get that our final solution will be:

$$y_B(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi at}{L}\sin\frac{n\pi x}{L},$$

with

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}dx.$$

Recall $g(x)$ is the initial velocity of the string $y_t(x,0) = g(x)$.

Our final solution will be $y(x,t) = y_A(x,t) + y_B(x,t)$.

Example - Suppose we have a string of length L whose initial velocity is 0, and whose initial position is given by

$$f(x) = \left\{ \begin{array}{cc} bx & 0 \leq x \leq L/2 \\ b(L-x) & L/2 \leq x \leq L \end{array} \right. .$$

Find $y(x,t)$ for this system.

Solution - The nth Fourier sine coefficient of $f(x)$ is

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$= \frac{2}{L}\int_0^{L/2} bx\sin\frac{n\pi x}{L}dx + \frac{2}{L}\int_{L/2}^L b(L-x)\sin\frac{n\pi x}{L}dx.$$

If we evaluate this integral we get:

$$A_n = \frac{4bL}{n^2\pi^2} \sin \frac{n\pi}{2}.$$

So, our solution will be the series

$$y(x,t) = \frac{4bL}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}.$$

The d'Alembert Form

OK, now we've learned the Fourier series way of solving the wave equation. Let's learn the easy way. Suppose the initial velocity of our string is $y_t(x,0) = 0$, and the initial position is given by $y(x,0) = f(x)$. Then a long time ago a smart guy named d'Alembert figured out that, if $F(x)$ is the *odd* extension of period $2L$ of the initial position function $f(x)$, then

$$y(x,t) = \frac{1}{2}[F(x+at) + F(x-at)]$$

satisfies our differential equation. Check it out, it works. So, what this looks like is our initial position function breaks up into two equal halves, and one half travels one way down the string, while the other half travels the other way.

We can use some trigonometric substitutions (done in the book) to prove our Fourier series solution boils down to the d'Alembert form, or we can just stop to admire it's simplicity. I vote for the latter.