Math 2280 - Lecture 40

Dylan Zwick

Spring 2013

In today's lecture, the first of two on this section, we'll discuss how Fourier series can be used to solve a simple, but very important *partial* differential equation. Namely, the one-dimensional heat equation.

This lecture corresponds with section 9.5 from the textbook. The assigned problems are:

Section 9.5 - 1, 3, 5, 7, 9

Heat Conduction and Separation of Variables

The flow of heat through a long, thin rod can be modeled by the *one-dimensional heat equation*:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Here, $u(x,t)$ is a function of both displacement, x, and time, t, and k is a given positive constant called the *thermal diffusivity*.

We want to solve this equation for a given set of *boundary conditions*. In ordinary differential equations, the boundary conditions are usually numbers. In partial differential equations, the boundary conditions are usually functions. Here we'll assume our boundary conditions are of the form:

$$u(0,t) = u(L,t) = 0, \; (t > 0),$$
$$u(x,0) = f(x), \; (0 < x < L).$$

The important idea here is that our partial differential equation is *linear*. So, for any two solutions u_1, u_2 we have that $c_1 u_1 + c_2 u_2$ satisfy the partial differential equation, and if u_1, u_2 satisfy the above boundary conditions on x (called *homogeneous* boundary conditions) then u_1, u_2 will as well. Superposition does *not* work for the boundary condition $u(x,0) = f(x)$, and here is where we need Fourier series. We want to find a linear combination of our almost solutions such that at time $t = 0$ the linear combination is equal to $f(x)$, and gives us a solution.

Example - It is easy to verify by direction substitution that each of the functions:

$$u_1(x,t) = e^{-t}\sin x, \; u_2(x,t) = e^{-4t}\sin 2x, \; u_3(x,t) = e^{-9t}\sin 3x,$$

satisfy the equation $u_t = u_{xx}$. Use these functions to construct a solution to the boundary value problem with boundary values:

$$u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = 80\sin^3 x = 60\sin x - 20\sin 3x.$$

Solution - All our functions satisfy the boundary conditions $u(0,t) = u(\pi,t) = 0$, and so we want a linear combination such that:

$$c_1 e^{-t}\sin x + c_2 e^{-4t}\sin 2x + c_3 e^{-9t}\sin 3x = 60\sin x - 20\sin 3x$$

when $t = 0$. But this is easy. We can just eyeball it to get $c_1 = 60, c_2 = 0$, and $c_3 = -20$. So, our solution is:

$$u(x,t) = 60e^{-t}\sin x - 20e^{-9t}\sin 3x.$$

They won't all be this easy.

That last one was pretty easy. It's also the exception. Usually, we have to find an infinite number of solutions, and make an infinite series equal to $f(x)$. You knew it couldn't be that easy, right?

Separation of Variables

Suppose we have the boundary values $u(x, 0) = u(x, L) = 0$. We're going to assume our function $u(x, t)$ can be written as the product of two functions, one a function of x alone, and the other a function of t alone. This approach is called *separation of variables*. So,

$$u(x, t) = X(x)T(t).$$

Plugging this into our differential equation and doing some algebra we get

$$\frac{X''}{X} = \frac{T'}{kT}.$$

If both X and T are non-trivial functions, this is only possible if both are equal to a constant:

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda.$$

This gives us two *ordinary* differential equations. We're now back to familiar territory.

$$X'' + \lambda X = 0,$$
$$T' + \lambda kT = 0.$$

The first must satisfy the boundary conditions $X(0) = X(L) = 0$, and so we have an eigenvalue problem like the ones we dealt with in section 3.8.[1] Well, if we recall section 3.8, we'll remember that the allowable values of λ are

$$\lambda_n = \frac{n^2\pi^2}{L^2},$$

and the eigenfunctions are

$$X_n(x) = \sin\frac{n\pi x}{L}.$$

If we plug this value for λ into our differential equation for T we get:

$$T_n' + \frac{n^2\pi^2 k}{L^2}T_n = 0,$$

A non-trivial solution to this differential equation is:

$$T_n(t) = e^{-n^2\pi^2 kt/L^2}.$$

So, our solution will be:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L}.$$

We just need to determine what the coefficients c_n are. This ain't so bad. We want to pick the c_n so that they satisfy

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x).$$

[1]Bet you thought you were done with those, didn't you?

But this is the Fourier sine series for $f(x)$ on the interval $0 < x < L$, and so we have:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

And we've got our solution! Hooray!

Example - Suppose that a rod of length $L = 50cm$ is immersed in steam until its temperature is $u_0 = 100°C$ throughout. At time $t = 0$, its lateral surface is insulated and its two ends are imbedded in ice at $0°C$. Calculate the rod's temperature at its midpoint after half an hour if it is made of (a) iron ($k = .15$); (b) concrete ($k = .005$).

Solution - The boundary value problem for the rod is given by:

$$u_t = ku_{xx},$$
$$u(0,t) = u(L,t) = 0,$$
$$u(x,0) = u_0.$$

Now, we've solved the Fourier series for a square wave a bunch of times, so I'll just cut to the chase and give that the Fourier coefficients are

$$b_{2n+1} = \frac{4u_0}{(2n+1)\pi},$$

for the odd coefficients, and the even coefficients are 0. So, the temperature in the rod will be:

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n\ odd}^{\infty}\frac{1}{n}\left(e^{-\frac{n^2\pi^2k}{L^2}t}\right)\sin\left(\frac{n\pi x}{L}\right).$$

Plugging in $u_0 = 100, L = 50$, and $k = .15$ (for iron) we get that $u(25,1800) \approx 43.85°C$. Doing the same with $k = .005$ (for concrete) we get $u(25,1800) \approx 100.00°C$. So, concrete is a *very* good insulator.